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An Inverse Source Problem with Sparsity Constraint for the Time-Fractional Diffusion Equation

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Abstract. In this paper, an inverse source problem for the time-fractional diffusion equation is investigated. The observational data is on the final time and the source term is assumed to be temporally independent and with a sparse structure. Here the sparsity is understood with respect to the pixel basis, i.e., the source has a small support. By an elastic-net regularization method, this inverse source problem is formulated into an optimization problem and a semismooth Newton (SSN) algorithm is developed to solve it. A discretization strategy is applied in the numerical realization. Several one and two dimensional numerical examples illustrate the efficiency of the proposed method.

AMS subject classifications: 65N21, 49M15

Key words: Inverse source problem, time-fractional diffusion equation, sparse constraint, elastic-net regularization method, semismooth Newton method.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^d$, $d=1,2,3$ be an open bounded domain with C^1-boundary, we consider the following time-fractional diffusion equation with homogeneous Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\Delta u+f(x), & (x,t)\in\Omega\times(0,T), \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T], \\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

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The fractional derivative $\partial^\alpha u(x,t)/\partial t^\alpha$ is the Caputo fractional derivative which is defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\eta)^{-\alpha}\frac{\partial u}{\partial \eta}\mathrm{d}\eta, \quad 0<\alpha<1, \tag{1.2}$$

where $\Gamma(1-\alpha)$ is the Gamma function. The time-fractional diffusion equation has been successfully applied in many fields. For instance, in [24] it is applied to describe the diffusion in fractional geometry. The time-fractional diffusion is also closely related to a non-Markovian diffusion process [22] or continuous time random walks on fractals [27]. A comprehensive review on it can be found in [3], see also [15,25].

In last decades, the mathematical analysis and the numerical realization to the time-fractional diffusion equation have been studied in many literatures, see e.g., [4,10,12,18,19,23,28] and references cited there. Meanwhile, many scholars consider the inverse problem corresponding to time-fractional diffusion equation. For example, based on the eigenfunction expansion and the Gel'fand-Levitan theory, the uniqueness of identifying the order of the fractional derivative and diffusion coefficient was established in Chen et al. [1] for one-dimensional time fractional diffusion equation. In [14], Jin and Rundell proposed an algorithm of the quasi-Newton type to reconstruct a spatially varying potential term in a one-dimensional time-fractional diffusion equation and the unique identifiability of the inverse problem had been established in the case where the time is sufficiently large and the set of input sources forms a complete basis in $L^2(0,1)$. Liu and Yamamoto proposed a numerical scheme for the backward problem based on the quasi-reversibility method in [20] and derived error estimates for the approximation under a priori smoothness assumption on the initial condition. In [34], Wang and Liu applied the data regularizing technique to deal with the backward problem, under the a priori information about the bound on initial function, the Hölder convergence result was established. An regularization method was proposed to solve a time fractional order backward heat conduction problem and the optimal stability error estimation was obtained in Xiong et al. [37]. Ye and Xu proposed a time-space spectral approximation algorithm based on the optimal control framework to solve the backward problem and they obtained a priori error estimate for the spectral approximation in [38]. By applying the separation of variables, Wang et al. [33] reconstructed a space-dependent source for the time-fractional diffusion equation by Tikhonov regularization method and provided the convergence estimates under an a priori and a posteriori parameter choice rule. In [36], Wei and Zhang transformed the time-dependent inverse source problem into a first kind Volterra integral equation and used a boundary element method combined with a generalized Tikhonov regularization to solve the Volterra integral equation. Based on the method of the eigenfunction expansion, the uniqueness of the inverse problem was proved by analytic continuation and Laplace transform in Zhang and Xu [39]. Zheng and Wei considered Fourier regularization method to solve the sideway problem for the time fractional advection-dispersion equation in a quarter plane in [40] and they obtained the convergence under a priori bound assumptions for the exact solution.