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A Simple Implementation of the Semi-Lagrangian Level-Set Method

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Abstract. Semi-Lagrangian (S-L) methods have no CFL stability constraint, and are more stable than the Eulerian methods. In the literature, the S-L method for the level-set re-initialization equation was complicated, which may be unnecessary. Since the re-initialization procedure is auxiliary, we propose to use the first-order S-L scheme coupled with a projection technique to improve the accuracy at the grid points just adjacent to the interface. Standard second-order S-L method is used for evolving the level-set convection equation. The implementation is simple, including on the block-structured adaptive mesh. The efficiency of the S-L method is demonstrated by extensive numerical examples including passive convection of interfaces with corners/kinks/large deformation under given velocity fields, a geometrical flow with topological changes, simulations of bubble/ droplet dynamics in incompressible two-phase flows. In terms of accuracy it is comparable to the other existing methods.

AMS subject classifications: 65M06, 65M20, 76T10

Key words: Semi-Lagrangian method, level-set method, interface motion, two-phase flow, bubble/ droplet dynamics, block-structured adaptive mesh.

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1 Introduction

The level-set method of Osher and Sethian [11] has been an invaluable tool in computing interface problems for a wide range of applications.

The level-set convection equation reads

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0, \tag{1.1}$$

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where ϕ is the level-set function, its zero level-set being the interface. $\mathbf{u}$ is the convective velocity field.

In practice, the level-set function ϕ needs to be re-initialized to prevent the formation of too large or small gradient of ϕ around the interface. Originally proposed in [22], the following auxiliary Hamilton-Jacobi equation has been widely used and studied for the purpose of re-initialization:

$$\begin{cases} \dfrac{\partial \phi}{\partial \tau}+S(\phi_0)(|\nabla\phi|-1)=0, \\ \phi(\mathbf{x},0)=\phi_0(\mathbf{x}), \end{cases} \tag{1.2}$$

where ϕ_0 is the level-set function before the re-initialization, τ is the pseudo-time and $S(x)$ is the sign function of x defined as

$$S(x)=\begin{cases} -1, & \text{if } x<0, \\ 0, & \text{if } x=0, \\ 1, & \text{if } x>0. \end{cases} \tag{1.3}$$

As $\tau\to\infty$, the solution of (1.2) approaches to a steady state, i.e., the so-called signed distance function. In many practical applications, the re-initialization equation (1.2) does not need to be solved to the steady state. Instead, just a few time iterations are enough. Though the level-set function ϕ is one dimension higher than the interface, the computational cost can be reduced significantly by using the local level-set technique proposed in [13]. In the local level-set approach, the Eqs. (1.1) and (1.2) are solved only in small tubes containing the interface, since only the zero level set is physical.

The semi-Lagrangian (S-L) methods trace back along the characteristic curves to locate the departure points by solving ordinary differential equations, and then use appropriate interpolations to get the approximation of solution at the grid points. Generally speaking the S-L methods automatically satisfy the CFL condition by shifting the stencil. Thus it is more stable than the classical Eulerian methods. The S-L methods do not handle shock discontinuities well. However, there is no shock for the level-set function since it is at least Lipschitz continuous (e.g., [17]).

The S-L methods date back at least to the Courant-Isaacson-Rees (CIR) method [3]. They have been popular in numerical computations in atmospheric sciences (e.g., [16, 29]). The S-L methods have received a lot of interest in the level-set community recently. In a series of works [17–21], J. Strain studied the S-L schemes for the level-set equation (1.1), the computational geometry based method on quadtrees was used for the re-initialization. Adaptive mesh methods on tree-structured Cartesian grid were also developed. In [4,5,14], the first-order semi-Lagrangian scheme was used as building blocks and the differential equations were evolved forward and backward in time to get an error estimate, then the error information was exploited in another forward evolution step to obtain more accurate solution. In [7], a hybrid particle level-set method was proposed to improve the accuracy of the first-order S-L scheme. In [10], a non-graded adaptive mesh