Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 1, pp. 173-185

DOI: 10.4208/aamm.2015.m970
February 2017

Numerical Solution of Euler-Lagrange Equation with Caputo Derivatives

Tomasz Blaszczyk[1,*] and Mariusz Ciesielski[2]

[1] *Czestochowa University of Technology, Institute of Mathematics, al. Armii Krajowej 21, 42-201 Czestochowa, Poland*

[2] *Czestochowa University of Technology, Institute of Computer and Information Sciences, ul. Dabrowskiego 73, 42-201 Czestochowa, Poland*

Received 13 February 2015; Accepted (in revised version) 22 February 2016

Abstract. In this paper the fractional Euler-Lagrange equation is considered. The fractional equation with the left and right Caputo derivatives of order $\alpha \in (0,1]$ is transformed into its corresponding integral form. Next, we present a numerical solution of the integral form of the considered equation. On the basis of numerical results, the convergence of the proposed method is determined. Examples of numerical solutions of this equation are shown in the final part of this paper.

AMS subject classifications: 26A33, 34A08, 65R20, 70H03

Key words: Fractional Euler-Lagrange equation, fractional integral equation, numerical solution, Caputo derivatives.

1 Introduction

In the past few years, many applications in real phenomena have been found, where certain dynamics are described not only by integer but also by real order operators [4, 15, 16, 23, 29, 33, 34, 38]. An important issue is that the derivative of fractional order has a local property at any point of a domain only when order is an integer number. For non-integer cases, the fractional derivative is a nonlocal operator and depends on the past values of a function (left derivative) or future ones (right derivative). We refer the reader to a summary of fractional calculus theory in monographs [6, 18, 24, 27] and papers [1, 14, 19, 20, 26, 31, 36, 37] that cover various problems in this field.

One natural application of fractional operators is variational calculus. In this approach, one modifies the variational principle with replacing the integer order operators by a fractional one. Then, the minimisation of the action leads to the fractional differential

*Corresponding author.
Email: tomasz.blaszczyk@im.pcz.pl (T. Blaszczyk), mariusz.ciesielski@icis.pcz.pl (M. Ciesielski)

equations which are known in the literature as the fractional Euler-Lagrange equations. Different approaches have been considered in recent years e.g., the Lagrangian or Hamiltonian approach with fractional integrals or fractional derivatives [1, 3, 5, 7, 19, 21].

The main feature of the fractional Euler-Lagrange equations is that the fractional operator appearing in these equations contains simultaneously the left and right derivative. This is also a fundamental problem in finding solutions of equations of a variational type [6]. Consequently, numerical methods have been devoted to solving fractional variational problems [8–12, 28, 35].

In this paper we present a numerical solution of the Euler-Lagrange equation with Caputo derivatives in the finite time interval.

2 Fractional preliminaries

In this section, we introduce the fractional derivatives and integrals used in this work and some of their properties (see [18, 25, 27]). The left and right Caputo derivatives of order $\alpha \in (0,1]$ are defined as follows

$$ {}^C D^{\alpha}_{a^+} x(t) := I^{1-\alpha}_{a^+} D x(t), \tag{2.1a} $$

$$ {}^C D^{\alpha}_{b^-} x(t) := -I^{1-\alpha}_{b^-} D x(t), \tag{2.1b} $$

where D is an operator of the first order derivative and operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are the left and right fractional integrals of order $\alpha > 0$, respectively, defined by

$$ I^{\alpha}_{a^+} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \qquad (t > a), \tag{2.2a} $$

$$ I^{\alpha}_{b^-} x(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{x(\tau)}{(\tau-t)^{1-\alpha}} d\tau, \qquad (t < b). \tag{2.2b} $$

If $\alpha = 1$, then ${}^C D^1_{a^+} x = x'$ and ${}^C D^1_{b^-} x = -x'$.

The composition rules of the fractional operators (for $\alpha \in (0,1]$) are as follows [18, 22]

$$ I^{\alpha}_{a^+} {}^C D^{\alpha}_{a^+} x(t) = x(t) - x(a), \tag{2.3a} $$

$$ I^{\alpha}_{b^-} {}^C D^{\alpha}_{b^-} x(t) = x(t) - x(b), \tag{2.3b} $$

and the fractional integral of a constant C

$$ I^{\alpha}_{a^+} C = C \frac{(t-a)^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.4} $$