Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 1, pp. 186-204

DOI: 10.4208/aamm.2014.m63
February 2017

A Quadratic Triangular Finite Volume Element Method for a Semilinear Elliptic Equation

Zhiguang Xiong* and Kang Deng

School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, China

Received 21 December 2014; Accepted (in revised version) 5 April 2016

Abstract. In this paper we extend the idea of interpolated coefficients for a semilinear problem to the quadratic triangular finite volume element method. At first we introduce quadratic triangular finite volume element method with interpolated coefficients for a boundary value problem of semilinear elliptic equation. Next we derive convergence estimate in H^1-norm, L^2-norm and L^∞-norm, respectively. Finally an example is given to illustrate the effectiveness of the proposed method.

AMS subject classifications: 65N30

Key words: Semilinear elliptic equation, triangulation, finite volume element with interpolated coefficients.

1 Introduction

The finite volume element method is a discretization technique for solving partial differential equations, especially for those that arise from physical laws including mass, momentum, and energy. The method has been widely used in computational fluid mechanics and other applications because it keeps the mass conservation [2, 5–7, 11, 12, 14, 15, 17, 18, 21, 22, 25–28, 34]. As far as the method is concerned, it is identical to the special case of the generalized difference method or GDM proposed by Li-Chen-Wu [21].

The finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal [35]. Later Larsson-Thomee-Zhang [19] studied the semidiscrete linear triangular finite element with interpolated coefficients and Chen-Larsson-Zhang [10] derived almost optimal order convergence on piecewise uniform triangular meshes by use of superconvergence techniques. Xiong-Chen studied superconvergence of finite element for some semilinear elliptic problems [29–31]. Xiong-Chen first put the interpolation idea

*Corresponding author.
Email: zgxiong@hnust.edu.cn (Z. G. Xiong), kdeng@hnust.edu.cn (K. Deng)

into the finite volume element method and studied the finite volume element with interpolated coefficients of the two-point boundary problem [32] and the linear triangular finite volume element method for a class of semilinear elliptic equations [33].

Li [20] has considered the finite volume element method for a nonlinear elliptic problem and obtained the error estimate in H^1-norm. Chatzipantelidis-Ginting-Lazarov [8] have studied the finite volume element method for a nonlinear elliptic problem, established the error estimates in H^1-norm, L^2-norm and L^∞-norm. Bi [3] obtains the H^1 and $W^{1,\infty}$ superconvergence estimates between the solution of the finite volume element method and that of the finite element method for a nonlinear elliptic problem. In this paper, we put the excellent interpolating coefficients idea into the finite volume element method on triangular mesh for a semilinear elliptic equation.

We denote Sobolev space and its norm by $W^{k,r}(\Omega)$ and $\|\cdot\|_{k,r}$, respectively [1]. If $r=2$, simply use $H^k(\cdot)$ and $\|\cdot\|_k$ and $\|\cdot\|=\|\cdot\|_0$ is L^2-norm. Further we denote with r' the adjoint of r, i.e.,

$$\frac{1}{r}+\frac{1}{r'}=1, \quad r\geq 1.$$

We assume that the exact solution u is sufficiently smooth for our purpose. Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h.

The rest of the paper is organized as follow. First we introduce the quadratic triangular finite volume element method with interpolated coefficients in Section 2 and give preliminaries and some lemmas in Section 3. Next we derive optimal order H^1-norm, L^2-norm and L^∞-norm estimates, respectively, in Section 4. Finally the theoretical results are tested by a numerical example in Section 5.

2 Quadratic finite volume element method with interpolated coefficients

Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal domain. Consider the second-order semilinear elliptic boundary value problem:

$$\begin{cases} -\Delta u+f(u)=g & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is assumed that $f(s)$ is the sufficiently smooth function with respect to s, and $f'(s)>0$ for finite interval.

Let $V\subset\Omega$ be any control volume with piecewise smooth boundary ∂V. Integrate (2.1) over control volume V, then by the Green's formula, the conservative integral of (2.1) reads, finding u, such that

$$-\int_{\partial V}\frac{\partial u}{\partial n}\mathrm{d}s+\int_V f(u)\mathrm{d}x\mathrm{d}y=\int_V g\mathrm{d}x\mathrm{d}y, \quad V\subset\Omega. \tag{2.2}$$

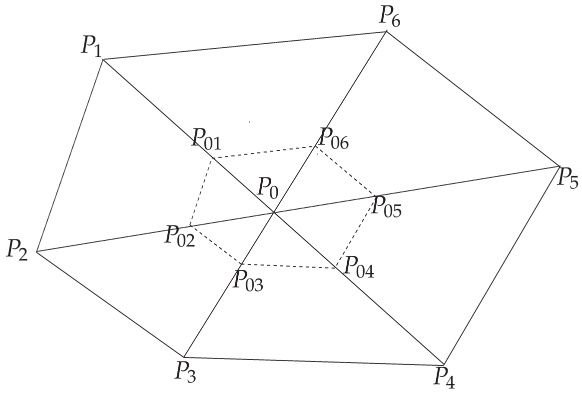

Figure 1: Illustration for a dual element V_{P_0} and its modes.

The FVE method of (2.2) consists of replacing by finite-dimensional space of piecewise smooth function and using a finite set of volumes. In this paper, we shall consider triangular partition of Ω and piecewise quadratic triangle interpolation with interpolated coefficients, for u.

Give a quasi-uniform triangulation $\mathcal{J}_h$ for Ω with $h=\max h_K$, where h_K is the diameter of the triangle $K\in\mathcal{J}_h$. Let Q_K be the barycentre of $K\in\mathcal{J}_h$. The vertexes of the triangles and the midpoints of the sides are taken as the nodes. $\bar{\Omega}_h$ denotes the set of the vertexes of all the triangular elements, $\bar{M}_h$ the set of the midpoints of the sides of all elements. All the control volumes constitute the dual partition $\mathcal{J}_h^*$, consisting of the polygons $K^*_{P_0}$ surrounding the node $P_0\in\bar{\Omega}_h$ and K^*_M surrounding $M\in\bar{M}_h$. Their detailed construction is as follows [21]:

1) Construction of $K^*_{P_0}$. Suppose that $P_0\in\bar{\Omega}_h$, that P_i are its adjacent vertexes, and that P_{0i} is a point on $\overline{P_0P_i}$ such that $\overline{P_0P_{0i}}=\frac{1}{3}\overline{P_0P_i}$. Connect successively P_{0i} to obtain a polygon K_{P_0} (see Fig. 1).

2) Construction of K^*_M. Let $M\in\bar{M}_h$ be a midpoint of a common side of two adjacent triangular elements $K_{Q_1}=\triangle P_0P_1P_2$ and $K_{Q_2}=\triangle P_0P_1P_3$. Denote by Q_{12}, Q_{13}, Q_{02}, Q_{03} the midpoints of $\overline{P_{01}P_{02}}$, $\overline{P_{01}P_{03}}$, $\overline{P_{10}P_{12}}$ and $\overline{P_{10}P_{13}}$ respectively. A polygon K^*_M surround M is obtained by connecting successively P_{10}, Q_{03}, Q_2, Q_{13}, P_{01}, Q_{12}, Q_1, Q_{02}, P_{10} (see Fig. 2).

For boundary nodes, their control volumes should be modified correspondingly.

Let $S_h\subset H^1(\Omega)$ and $S_{0h}\subset H^1_0(\Omega)$ be both the piecewise quadratic triangular finite element subspace over the partition $\mathcal{J}_h$, and S_h^* be the piecewise constant space over the dual partition $\mathcal{J}_h^*$. Denote ϕ_P by basic function of S_h at the node $P\in\bar{\Omega}_h\cup\bar{M}_h$. For an arbitrary node $P\in\bar{\Omega}_h\cup\bar{M}_h$, denote χ_P or χ_M by characteristic function over V_P or V_M. Define standard Lagrangian interpolation operator $\mathrm{I}_h: C(\Omega)\to S_h$ by

$$\mathrm{I}_h\varphi=\sum_{P\in\bar{\Omega}_h\cup\bar{M}_h}\varphi(P)\phi_P,\quad \forall\varphi\in C(\Omega), \tag{2.3}$$

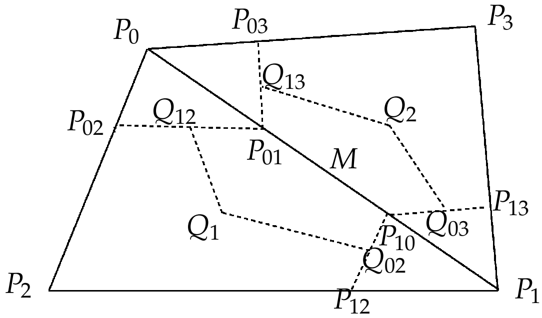

Figure 2: Illustration for a dual element V_M and its modes.

and interpolation operator $\mathrm{I}_h^*: C(\Omega) \to S_h^*$ by

$$\mathrm{I}_h^* \varphi = \sum_{P \in \bar{\Omega}_h \cup \bar{M}_h} \varphi(P) \chi_P, \quad \forall \varphi \in C(\Omega). \tag{2.4}$$

The standard finite volume element scheme of (2.2) can read, finding $\bar{u}_h \in S_{0h}$, such that

$$-\int_{\partial V_P} \frac{\partial \bar{u}_h}{\partial n} \mathrm{d}s + \int_{V_P} f(\bar{u}_h) \mathrm{d}x\mathrm{d}y = \int_{V_P} g \mathrm{d}x\mathrm{d}y, \quad \forall P \in \bar{\Omega}_h \cup \bar{M}_h.$$

For the sake of simplicity, we now define quadratic triangular finite volume element scheme with interpolated coefficients, finding $u_h \in S_{0h}$, such that

$$-\int_{\partial V_P} \frac{\partial u_h}{\partial n} \mathrm{d}s + \int_{V_P} \mathrm{I}_h f(u_h) \mathrm{d}x\mathrm{d}y = \int_{V_P} g \mathrm{d}x\mathrm{d}y, \quad \forall P \in \bar{\Omega}_h \cup \bar{M}_h. \tag{2.5}$$

Eq. (2.5) can be further written as difference equation which is simpler than that of standard finite volume element method [32]. Notice that $\mathrm{I}_h f(u_h) = \sum_{P \in \bar{\Omega}_h \cup \bar{M}_h} f(u_h(P)) \phi_P$ and one can be solved by the Newton iteration method in which its tangent matrix can be calculated simply.

3 Preliminaries and lemmas

In the preceding section, we give the finite volume element scheme with interpolated coefficients. We will give preliminary work and some lemmas in this section. Letting

$$a(u, \mathrm{I}_h^* \varphi_h) = -\sum_{P \in \bar{\Omega}_h \cup \bar{M}_h} \varphi_h(P) \int_{\partial V_P} \frac{\partial u}{\partial n} \mathrm{d}s, \qquad \forall \varphi_h \in S_{0h},$$

$$(u, \mathrm{I}_h^* \varphi_h) = \sum_{P \in \bar{\Omega}_h \cup \bar{M}_h} \varphi_h(P) \int_{V_P} u \mathrm{d}x\mathrm{d}y, \qquad \forall \varphi_h \in S_{0h},$$

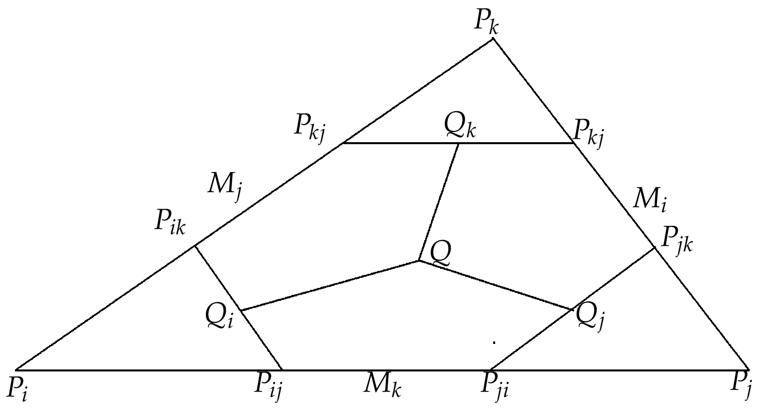

Figure 3:

and taking $V = V_P$, (2.2) can be written as, finding $u \in H_0^1(\Omega)$, such that

$$a(u, \mathrm{I}_h^* \varphi_h) + (f(u), \mathrm{I}_h^* \varphi_h) = (g, \mathrm{I}_h^* \varphi_h), \qquad \forall \varphi_h \in S_{0h}. \tag{3.1}$$

Analogously, (2.5) is equivalent to finding $u_h \in S_{0h}$, such that

$$a(u_h, \mathrm{I}_h^* \varphi_h) + (\mathrm{I}_h f(u_h), \mathrm{I}_h^* \varphi_h) = (g, \mathrm{I}_h^* \varphi_h), \qquad \forall \varphi_h \in S_{0h}. \tag{3.2}$$

For the sake of simplicity in our analysis in the paper, we still denote the bilinear form by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}y, \quad \forall u, v \in H_0^1(\Omega).$$

Depicted as in Fig. 3, we convert the integral on the edge of dual partition to the related element $K = \triangle P_i P_j P_k \in \mathcal{J}_h$, then

$$a(u, \mathrm{I}_h^* \varphi_h) = -\sum_{K \in \mathcal{J}_h} \sum_{l=i,j,k} \Big[\varphi_h(P_l) \int_{\partial V_{P_l} \cap K} \frac{\partial u}{\partial n} \mathrm{d}s + \varphi_h(M_l) \int_{\partial V_{M_l} \cap K} \frac{\partial u}{\partial n} \mathrm{d}s \Big], \quad \forall \varphi_h \in S_{0h}. \tag{3.3}$$

Similarly, we can obtain

$$\begin{aligned}(u, \mathrm{I}_h^* \varphi_h) &= \sum_{K \in \mathcal{J}_h} \int_K u \mathrm{I}_h^* \varphi_h \mathrm{d}x\mathrm{d}y \\ &= \sum_{K \in \mathcal{J}_h} \sum_{l=i,j,k} \Big[\varphi_h(P_l) \int_{V_{P_l} \cap K} u \mathrm{d}x\mathrm{d}y + \varphi_h(M_l) \int_{V_{M_l} \cap K} u \mathrm{d}x\mathrm{d}y \Big], \quad \forall \varphi_h \in S_{0h}. \end{aligned} \tag{3.4}$$

Denote $\|\cdot\|_s$ and $|\cdot|_s$ be continuous norm and continuous semi-norm of order s in Sobolev space $H^s(\Omega)$, respectively. Let us introduce the discrete zero norm, semi-norm and fullnorm, respectively, by

$$\|\varphi_h\|_{0,h}=\Big\{\sum_{K\in\mathcal{J}_h}\|\varphi_h\|^2_{0,h,K}\Big\}^{1/2}, \tag{3.5a}$$

$$|\varphi_h|_{1,h}=\Big\{\sum_{K\in\mathcal{J}_h}|\varphi_h|^2_{1,h,K}\Big\}^{1/2}, \tag{3.5b}$$

$$\|\varphi_h\|_{1,h}=\big(\|\varphi_h\|^2_{0,h}+|\varphi_h|^2_{1,h}\big)^{1/2}, \tag{3.5c}$$

for $\varphi_h\in S_{0h}$, where $K=\triangle P_iP_jP_k$, shown as in Fig. 3, and

$$\begin{aligned}\|\varphi_h\|_{0,h,K}&=\Big[\frac{1}{6}(\varphi^2_{P_i}+\varphi^2_{P_j}+\varphi^2_{P_k}+\varphi^2_{M_i}+\varphi^2_{M_j}+\varphi^2_{M_k})S_K\Big]^{1/2},\\ |\varphi_h|_{1,h,K}&=[(\varphi_{P_i}-\varphi_{M_i})^2+(\varphi_{P_j}-\varphi_{M_j})^2+(\varphi_{P_k}-\varphi_{M_k})^2\\ &\quad+(\varphi_{M_i}-\varphi_{M_j})^2+(\varphi_{M_j}-\varphi_{M_k})^2]^{1/2}.\end{aligned}$$

From [21], we have the following lemma.

Lemma 3.1. *For all $\varphi_h\in S_{0h}$, $|\varphi_h|_{1,h}$ and $|\varphi_h|_1$ are identical and $\|\varphi_h\|_{0,h}$ and $\|\varphi_h\|_{1,h}$ are equivalent with $\|\varphi_h\|_0$ and $\|\varphi_h\|_1$ respectively, i.e., there exist positive constants C_1, C_2, C_3, C_4 independent of h such that*

$$C_1|\varphi_h|_{0,h}\le|\varphi_h|_0\le C_2|\varphi_h|_{0,h},\qquad \forall\varphi_h\in S_h, \tag{3.6a}$$

$$C_3\|\varphi_h\|_{1,h}\le\|\varphi_h\|_1\le C_4\|\varphi_h\|_{1,h},\qquad \forall\varphi_h\in S_h. \tag{3.6b}$$

From [7,9,21], we have three lemmas.

Lemma 3.2. *There exist positive constants C_1, C_2 such that*

$$a(\varphi_h,\mathrm{I}^*_h\varphi_h)\ge C_1|\varphi_h|^2_1,\qquad \forall\varphi_h\in S_{0h}, \tag{3.7a}$$

$$|a(u-\mathrm{I}_hu,\mathrm{I}^*_h\varphi_h)|\le C_2h^2\|u\|_3|\varphi_h|_1,\qquad \forall u\in H^1_0(\Omega),\quad \varphi_h\in S_{0h}. \tag{3.7b}$$

Lemma 3.3. *The semi-norm $|\cdot|_1$ and the norm $\|\cdot\|_1$ are equivalent in the space $H^1_0(\Omega)$, that is, there exists a positive constants C such that*

$$|\varphi_h|_1\le\|\varphi_h\|_1\le C|\varphi_h|_1,\quad \forall\varphi_h\in S_{0h}. \tag{3.8}$$

Lemma 3.4. *The interpolation operator I^*_h has the following properties*

$$||\mathrm{I}^*_hv_h||_{e,\infty}\le||v_h||_{e,\infty},\qquad \forall v_h\in S_{0h}\ \text{for any side } e \text{ of } K\in\mathcal{J}_h, \tag{3.9a}$$

$$||\varphi_h-\mathrm{I}^*_h\varphi_h||_{0,p,K}\le Ch|\varphi_h|_{1,p,K},\qquad \forall\varphi_h\in S_{0h},\quad 1\le p\le\infty. \tag{3.9b}$$

In addition in [9], for the interpolation operator I_h, the following lemma is derived.

Lemma 3.5. *Assume that w, φ are sufficiently smooth functions. Let $\mathrm{I}_h\varphi \in S_{0h}$ be the Lagrangian interpolation of φ, then*

$$|(\varphi - \mathrm{I}_h\varphi, \psi_h)| \leq Ch^3 \|\varphi\|_{2,p} \|\psi_h\|_{1,p'}, \quad \forall \psi_h \in S_{0h}, \tag{3.10}$$

for

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p \leq \infty.$$

Lemma 3.6. *Assume $u \in H_0^1(\Omega)$, then there exists a positive constant C, independent of the mesh size h, such that*

$$|(u - \mathrm{I}_h u, \mathrm{I}_h^* \varphi_h)| \leq Ch^3 \|u\|_2 \|\varphi_h\|_0, \quad \forall \varphi_h \in S_{0h}. \tag{3.11}$$

Proof. In view of the Schwartz inequality, we easily give the desired (3.11). ☐

In addition in [7,8], the following lemma is derived.

Lemma 3.7. *Let e be a side of a triangle $K \in \mathcal{J}_h$. Then for $u \in H^1(K)$ there exists a constant $C > 0$ independent of h such that*

$$\Big| \int_e u(v_h - \mathrm{I}_h^* v_h) \mathrm{d}s \Big| \leq Ch^2 \|u\|_{1,K} \|v_h\|_{1,K}, \quad \forall v_h \in S_h. \tag{3.12}$$

Moreover, for $u \in H^1$ and $v_h \in S_{0h}$,

$$(u, v_h - \mathrm{I}_h^* v_h) \leq Ch^2 \|u\|_1 \|v_h\|_1. \tag{3.13}$$

For our theoretical analysis, we also need two lemmas as follows.

Lemma 3.8. *Let $u \in H^2$. The following identities hold*

$$\sum_{K \in \mathcal{J}_h} \int_{\partial K} \frac{\partial u}{\partial n} v_h \mathrm{d}s = 0, \qquad \sum_{K \in \mathcal{J}_h} \int_{\partial K} \frac{\partial u}{\partial n} \mathrm{I}_h^* v_h \mathrm{d}s = 0. \tag{3.14}$$

Proof. The first identity of (3.14) is obvious by rewriting the sum as integrals of jump terms over the interior edges of $\mathcal{J}_h$. These jumps obviously vanish because of the continuity of $\partial u / \partial n$. A similar argument gives the second identity of (3.14). ☐

Lemma 3.9. *It holds*

$$|a(u_h, v_h) - a(u_h, \mathrm{I}_h^* v_h)| \leq C(h^2 \|u_h\|_1 + h^2 \|u\|_2 + h \|u - u_h\|_1) \|v_h\|_1. \tag{3.15}$$

Proof. Using the Green's formula, the identity

$$\begin{aligned}&\int_{V_P\cap K}\Big(\frac{\partial}{\partial x}W_h^{(1)}+\frac{\partial}{\partial y}W_h^{(2)}\Big)\mathrm{d}x\mathrm{d}y\\=&\int_{V_P\cap\partial K}(W_h^{(1)},W_h^{(2)})\cdot n\mathrm{d}s+\int_{\partial V_P\cap K}(W_h^{(1)},W_h^{(2)})\cdot n\mathrm{d}s\end{aligned}\tag{3.16}$$

holds for $P\in Z_h^0$ and $K\in\mathcal{J}_h$, and hence we have

$$\begin{aligned}a(u_h,\mathrm{I}_h^*v_h)=&-\sum_{K\in\mathcal{J}_h}\int_K\Big(\frac{\partial}{\partial x}W_h^{(1)}+\frac{\partial}{\partial y}W_h^{(2)}\Big)\mathrm{I}_h^*v_h\mathrm{d}x\mathrm{d}y\\&+\sum_{K\in\mathcal{J}_h}\int_{\partial K}(W_h^{(1)},W_h^{(2)})\cdot n\mathrm{I}_h^*v_h\mathrm{d}s.\end{aligned}\tag{3.17}$$

By use of the Green's formula, we also obtain

$$\begin{aligned}&a(u_h,v_h)\\=&\sum_{K\in\mathcal{J}_h}\int_K\Big(W_h^{(1)}\frac{\partial v_h}{\partial x}+W_h^{(2)}\frac{\partial v_h}{\partial y}\Big)\mathrm{d}x\mathrm{d}y\\=&-\sum_{K\in\mathcal{J}_h}\int_K\Big(\frac{\partial}{\partial x}W_h^{(1)}+\frac{\partial}{\partial y}W_h^{(2)}\Big)v_h\mathrm{d}x\mathrm{d}y+\sum_{K\in\mathcal{J}_h}\int_{\partial K}(W_h^{(1)},W_h^{(2)})\cdot nv_h\mathrm{d}s.\end{aligned}\tag{3.18}$$

Subtracting (3.17) from (3.18) gives

$$\begin{aligned}a(u_h,v_h)-a(u_h,\mathrm{I}_h^*v_h)=&-\sum_{K\in\mathcal{J}_h}\int_K\Big(\frac{\partial}{\partial x}W_h^{(1)}+\frac{\partial}{\partial y}W_h^{(2)}\Big)(v_h-\mathrm{I}_h^*v_h)\mathrm{d}x\mathrm{d}y\\&+\sum_{K\in\mathcal{J}_h}\int_{\partial K}(W_h^{(1)},W_h^{(2)})\cdot n(v_h-\mathrm{I}_h^*v)\mathrm{d}s.\end{aligned}\tag{3.19}$$

Lemma 3.8 gives the identity

$$\sum_{K\in\mathcal{J}_h}\int_{\partial K}\Big(-W^{(1)}-(W_h^{(1)}-W^{(1)})_e,-W_h^{(2)}-(W_h^{(2)}-W_h^{(2)})_e\Big)\cdot n(v_h-\mathrm{I}_h^*v)\mathrm{d}s=0,$$

where

$$(W_h^{(i)}-W^{(i)})_e=a_{i1}(e)\frac{\partial u_h-u}{\partial x}+a_{i2}(e)\frac{\partial u_h-u}{\partial y},\quad i=1,2.$$

Employing this identity, (3.9) in Lemma 3.4, we get

$$
\begin{aligned}
&a(u_h,v_h)-a(u_h,\mathrm{I}_h^*v_h)\\
=&-\sum_{K\in\mathcal{J}_h}\int_K\Big(\frac{\partial}{\partial x}W_h^{(1)}-\xi_1+\frac{\partial}{\partial y}W_h^{(2)}-\xi_2\Big)(v_h-\mathrm{I}_h^*v_h)\mathrm{d}x\mathrm{d}y\\
&+\sum_{K\in\mathcal{J}_h}\int_{\partial K}\big((W_h^{(1)}-W^{(1)})-(W_h^{(1)}-W^{(1)})_e,(W_h^{(2)}-W_h^{(2)})\\
&-(W_h^{(2)}-W_h^{(2)})_e\big)\cdot n(v_h-\mathrm{I}_h^*v)\mathrm{d}s\\
\equiv&\sum_{K\in\mathcal{J}_h}(\mathrm{I}_K+\mathrm{II}_K),
\end{aligned}
\tag{3.20}
$$

where ξ_1 and ξ_2 are the mean values of $\frac{\partial}{\partial x}W_h^{(1)}$ and $\frac{\partial}{\partial y}W_h^{(2)}$ over triangle K, respectively. By using the Holder's inequality, we can get

$$
|\mathrm{I}_K|\le Ch(|W_h^{(1)}|_{1,K}+|W_h^{(2)}|_{1,K})h\|v_h\|_{1,K}\le Ch^2\|u_h\|_1\|v_h\|_{1,K}. \tag{3.21}
$$

To bound II_K, we have

$$
\begin{aligned}
|\mathrm{II}_K|\le&Ch\Big(\sum_{i=1}^{2}\Big|(a_{i1}-a_{i1}(e))\frac{\partial(u_h-u)}{\partial x}+(a_{i2}-a_{i2}(e))\frac{\partial(u_h-u)}{\partial y}\Big|_{1,K}\Big)\|v_h\|_{1,K}\\
\le&Ch\max|a_{ij}'|(\|u-u_h\|_{1,K}+h\|u\|_2)\|v_h\|_{1,K}.
\end{aligned}
\tag{3.22}
$$

Summing up (3.21) and (3.22) over all triangles, we obtain the desired (3.15). □

4 Error estimate of the finite volume element

We have given the definition of the finite volume element scheme with interpolated coefficients. Now we analyze the error of the scheme. To start our analysis, we introduce an auxiliary bilinear form

$$
A(u;w,\mathrm{I}_h^*\varphi_h)=a(w,\mathrm{I}_h^*\varphi_h)+(f'(u)w,\mathrm{I}_h^*\varphi_h),
$$

where u is the exact solution in (2.5). For the auxiliary bilinear form $A(u;\cdot,\cdot)$, we have following positive definite property.

Lemma 4.1. *Assume $f'(s)>0$ for finite interval, then for fixed $u\in H_0^1(\Omega)$, $A(u;w_h,\mathrm{I}_h^*w_h)$ is positive definite for sufficiently small h, i.e., there exists a positive constant α, such that*

$$
A(u;w_h,\mathrm{I}_h^*w_h)\ge\alpha(u,f)\|w_h\|_1^2,\quad\forall w_h\in S_{0h}. \tag{4.1}
$$

Proof. Rewrite $A(u;w_h,\mathrm{I}_h^*w_h)$ as

$$A(u;w_h,\mathrm{I}_h^*w_h)=a(w,\mathrm{I}_h^*w_h)+(f'(u_h)w_h,w_h)-[(f'(u)w_h,w_h)-(f'(u)w_h,\mathrm{I}_h^*w_h)]. \tag{4.2}$$

Application of Lemma 3.2 and Lemma 3.3 yields

$$a(w_h,\mathrm{I}_h^*w_h)\geq C_1\|w_h\|_1^2. \tag{4.3}$$

Note that $f'(s)>0$ and let $C_2=\inf_{P\in\Omega}f'(u_h(P))$ for the fixed u_h, then we have

$$(f'(u_h)w_h,w_h)\geq C_2\|w_h\|_0^2\geq 0. \tag{4.4}$$

In terms of (3.9) in Lemma 3.7, we obtain

$$\begin{aligned}
&|(f'(u_h)w_h,w_h)-(f'(u_h)w_h,\mathrm{I}_h^*w_h)|\\
=&\Big|\sum_{K\in\mathcal{J}_h}\int_K f'(u_h)w_h(w_h-\mathrm{I}_h^*w_h)\mathrm{d}x\mathrm{d}y\Big|\\
\leq&\sum_{K\in\mathcal{J}_h}Ch|f'(u_h)w_h|_{1,K}h|w_h|_{1,K}\\
\leq&\max_{\Omega}(|f''(u_h)\nabla u_h|,|f'(u_h)|)\sum_{K\in\mathcal{J}_h}Ch^2\|w_h\|_{1,K}^2\\
\leq&C_3h^2\|w_h\|_1^2.
\end{aligned} \tag{4.5}$$

Together (4.3), (4.4) with (4.5) yields

$$A(u_h;w_h,\mathrm{I}_h^*w_h)\geq C_1\|w_h\|_1^2-C_3h^2\|w_h\|_1^2=(C_1-C_3h^2)\|w_h\|_1^2,$$

which implies the desired result (4.1) for sufficiently small h. □

Lemma 4.2. *Assume $w\in H_0^1(\Omega)$, then for fixed $u_h\in S_{0h}$ there exists a positive constant C, independent of the mesh size h, such that*

$$|A(u_h;w-\mathrm{I}_hw,\mathrm{I}_h^*\varphi_h)|\leq Ch^2\|w\|_3\|\varphi_h\|_1,\quad \forall\varphi_h\in S_{0h}. \tag{4.6}$$

Proof. Rewrite $A(u_h;w-\mathrm{I}_hw,\mathrm{I}_h^*\varphi_h)$ as

$$\begin{aligned}
A(u_h;w-\mathrm{I}_hw,\mathrm{I}_h^*\varphi_h)=&a(w-\mathrm{I}_hw,\mathrm{I}_h^*\varphi_h)+(f'(u_h)(w-\mathrm{I}_hw),\varphi_h)\\
&+(f'(u_h)(w-\mathrm{I}_hw),\mathrm{I}_h^*\varphi_h)-(f'(u_h)(w-\mathrm{I}_hw),\varphi_h).
\end{aligned} \tag{4.7}$$

Again application of Lemma 3.2 and Lemma 3.3 yields

$$|a(w-\mathrm{I}_hw,\mathrm{I}_h^*\varphi_h)|\leq C^2\|w\|_3\|\varphi_h\|_1. \tag{4.8}$$

By Lemma 3.5, we obtain

$$|(f'(u_h)(w-\mathrm{I}_hw),\varphi_h)|\leq Ch^3\|w\|_3\|\varphi_h\|_1. \tag{4.9}$$

Recall Lemma 3.4,we also obtain the following inequality

$$\begin{aligned}
&|(f'(u_h)(w-\mathrm{I}_h w),\mathrm{I}_h^*\varphi_h)-(f'(u_h)(w-\mathrm{I}_h w),\varphi_h)| \\
\le&\sum_{K\in\mathcal{J}_h}\|f'(u_h)(w-\mathrm{I}_h w)\|_{0,K}\|\varphi_h-\mathrm{I}_h^*\varphi_h\|_{0,K} \\
\le&\max(f'(u_h))\sum_{K\in\mathcal{J}_h}\|w-\mathrm{I}_h w\|_{0,K}\|\varphi_h-\mathrm{I}_h^*\varphi_h\|_{0,K} \\
\le&Ch^4\|w\|_2\|\varphi_h\|_1.
\end{aligned} \tag{4.10}$$

Together (4.8), (4.9) with (4.10) yields the desired results (4.6). ☐

Now we state the main result of this section.

Theorem 4.1. *Assume $f'(s)>0$, $f\in C^2(R)$, $g\in L^2(\Omega)$. Let $u\in H_0^1(\Omega)\cap H^2(\Omega)$ be the solution of* (2.1) *and $\mathcal{J}_h$ be quasi-uniformly triangular partition of domain Ω, then the approximate solution $u_h\in S_{0h}$ of finite volume element method* (2.5) *with interpolated coefficients converges to the exact solution u with the following estimate*

$$\|u-u_h\|_1\le Ch^2. \tag{4.11}$$

Proof. Subtracting (3.2) from (3.1), we obtain the following error equation

$$a(u-u_h,\mathrm{I}_h^*\varphi_h)+(f(u)-\mathrm{I}_h f(u_h),\mathrm{I}_h^*\varphi_h)=0. \tag{4.12}$$

By expansion, we have

$$\begin{aligned}
f(u)-f(u_h)=&f'(u_h)(u-u_h)+(u-u_h)^2\int_0^1 f''(u_h-t(u_h-u))(1-t)\mathrm{d}t \\
\equiv&f'(u_h)(u-u_h)+(u-u_h)^2\bar{f}''.
\end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.12), we find

$$\begin{aligned}
&A(u_h;u_h-\mathrm{I}_h u_h,\mathrm{I}_h^*\varphi_h) \\
=&A(u_h;u-\mathrm{I}_h u,\mathrm{I}_h^*\varphi_h)+((u-u_h)^2\bar{f}''+f(u_h)-\mathrm{I}_h f(u_h),\mathrm{I}_h^*\varphi_h)\alpha\|\theta\|_1^2 \\
\le&(C\|u-\mathrm{I}_h u\|_1+C\|f(u_h)-\mathrm{I}_h f(u_h)\|+C\|(u-u_h)^2\|)\|\theta\|_1.
\end{aligned}$$

Application of Lemma 4.1, Lemma 4.2 and Lemma 3.6 yields

$$\|\theta\|_1\le Ch^2+C\|(u-u_h)^2\|. \tag{4.14}$$

By use of the property of the interpolation I_h, we obtain

$$\|(u-u_h)^2\|\le 2\|(u-\mathrm{I}_h u)^2\|_1+2\|\theta^2\|_1\le Ch^4+2\|\theta^2\|.$$

Substituting this into (4.14) yields

$$\|\theta\|_1 \leq Ch^2 + C\|\theta^2\|, \tag{4.15}$$

where the constants are dependent of u, u_h, f, g. Recalling for Bramble [4] that

$$\|\theta\|_{0,\infty} \leq C|\ln h|^{1/2}\|\nabla\theta\| \leq C|\ln h|^{1/2}\|\theta\|_1$$

holds for $\theta \in S_{0h}$, we get

$$\begin{aligned}\|\theta^2\| &= \Big(\int_\Omega \theta^4 \mathrm{d}x\mathrm{d}y\Big)^{1/2} \leq \max_\Omega|\theta|\Big(\int_\Omega \theta^2 \mathrm{d}x\mathrm{d}y\Big)^{1/2} \\ &= \|\theta\|_{0,\infty}\|\theta\| \leq C|\ln h|^{1/2}\|\theta\|_1\|\theta\| \leq C|\ln h|^{1/2}\|\theta\|_1^2.\end{aligned}$$

Substituting this into (4.15) yields

$$\|\theta\|_1 \leq C_1 h^2 + C_2|\ln h|^{1/2}\|\theta\|_1^2. \tag{4.16}$$

Now adopting a continuity argument by imitating the method by Frehse-Rannacher [16], we show

$$\|\theta\|_1 \leq \|\mathrm{I}_h u - u_h\|_1 \leq 2C_1 h^2. \tag{4.17}$$

For $s \in [0,1]$ considering the auxiliary semilinear elliptic problems (P^s): Find u^s such that

$$-\Delta u^s + sf(u^s) = sg \quad \text{in } \Omega, \quad u^s = 0 \quad \text{on } \partial\Omega. \tag{4.18}$$

Obviously, for $s=1$ this is our original problem (2.1) and for $s=0$ we have $u^0 \equiv 0$ on $\bar{\Omega}$. We shall assume the following condition on Ω. For any $s \in [0,1]$, there is a solution u^s of problem (P^s) and there is a constant Γ such that set

$$N_\Gamma = \Big\{\omega \,\Big|\, \omega \in H^2(\Omega) \cap H_0^1(\Omega),\ \max_\Omega |u-\omega| < \Gamma\Big\}$$

is some neighborhood of exact solution u in (2.1).

We approximate problem (P^s) by the discrete problems (P_h^s): Find $u_h^s \in S_{0h}$ such that

$$a(u_h^s, \mathrm{I}_h^* v_h) + s(\mathrm{I}_h f(u_h^s), \mathrm{I}_h^* v_h) = s(g, \mathrm{I}_h^* v_h), \quad \forall v_h \in S_{0h}. \tag{4.19}$$

We intend to show that (P_h^s) is solvable. For each h, we define the set $E_h \subset [0,1]$ by

$$E_h = \{s \in [0,1] \,|\, (\mathrm{P}_h^s) \text{ has a solution } u_h^s \in N_\Gamma \text{ and there holds } \|\mathrm{I}_h u^s - u_h^s\|_1 \leq 2C_1 h^2\},$$

where C_1 is the constant appearing in (4.16).

(i) E_h is not empty. In fact, for $s=0, u^s=0$ and $u_h^s=0$ are the solutions of continuous and the discrete problem, respectively.

(ii) E_h is open in $[0,1]$. In fact, if $s \in E_h$ then (P_h^s) is solvable and using the monotonicity condition, we obtain the solvability of (P_h^s) for all t in a neighborhood of s via the implicit function theorem. By the implicit function theorem u_h^t depends continuously on t. Thus properly shorten the neighborhood such that the strict inequality $\|\mathrm{I}_h u^s - u_h^s\|_1 < 2C_1h^2$ and $u_h^s \in N_\Gamma$ is still valid and we have $t \in E_h$ for these t.

(iii) E_h is closed. Let $s(j) \in E_h$ and $s(j) \to s, j \to \infty$. Since $u_h^{s(j)} \in N_\Gamma$ there is a cluster point u_h^s which is the unique solution of (P_h^s) and satisfies $\|\mathrm{I}_h u^s - u_h^s\|_1 \le 2C_1h^2$. Recalling for (4.16) we conclude

$$\|\mathrm{I}_h u^s - u_h^s\|_1 \le C_1h^2 + 4C_2C_1^2|\ln h|^{1/2}h^4 \le C_1(1+4C_1C_2|\ln h|^{1/2}h^2)h^2,$$

then for $h \le h^*(C_1,C_2)$, we have $4C_1C_2|\ln h|^{1/2}h^2 < 1$ and $\|\mathrm{I}_h u^s - u_h^s\|_1 < 2C_1h^2$, i.e., the strict inequality.

From (i)-(iii), we know that for $h \le h^*(C_1,C_2)$ the set E_h is not empty, closed and open with respect to $s \in [0,1]$ and thus must coincide with $[0,1]$. Note that for $s=1$, (P_h^1) is solvable. We prove that inequality (4.17) and $u_h \in N_\Gamma$ hold for appropriately small h.

Finally, the desired estimate (4.11) follows from (4.17) and the interpolation property

$$\|u - \mathrm{I}_h u\|_1 \le Ch^2\|u\|_2.$$

Thus, we complete the proof. □

For the proof of the L^2-norm estimate we shall employ a duality argument as the one used in [7,13], Let us consider the another auxiliary problem. Find $\varphi \in H_0^1$, such that

$$a(\varphi,v) + (f'(u_h)\varphi,v) = (u-u_h,v), \quad \forall v \in H_0^1. \tag{4.20}$$

Then the solution of (4.20) satisfies the following elliptic regularity estimate

$$\|\varphi\|_2 \le C\|u-u_h\|. \tag{4.21}$$

Theorem 4.2. *Assume $f'(s)>0$, $f \in C^2(R)$, $g \in L^2(\Omega)$, and $\mathcal{J}_h$ is quasi-uniform triangular partition of domain Ω. Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of* (2.1) *and $u_h \in S_{0h}$ be the approximate solution of finite volume element method* (2.5) *with interpolated coefficients, respectively. Then*

$$\|u-u_h\| \le C(u,f,g)h^3. \tag{4.22}$$

Proof. Firstly we note the following Taylor expansions

$$f(u) - f(u_h) = (u-u_h)\int_0^1 f'(u - t(u-u_h))\mathrm{d}t \equiv (u-u_h)\bar{f}', \tag{4.23a}$$

$$\begin{aligned} f(u) - f(u_h) - f'(u_h)(u-u_h) &= (u-u_h)^2 \int_0^1 f''(u_h - t(u_h-u))(1-t)\mathrm{d}t \\ &\equiv (u-u_h)^2\bar{f}''. \end{aligned} \tag{4.23b}$$

Then, in view of (4.20), we have

$$\begin{aligned}\|u-u_h\|^2 &= a(u-u_h,\varphi)+(f'(u_h)(u-u_h),\varphi)\\ &= a(u-u_h,\varphi)+(f(u)-f(u_h),\varphi)+(f(u_h)-f(u)+f'(u_h)(u-u_h),\varphi)\\ &= \big\{a(u-u_h,\varphi-\mathrm{I}_h\varphi)+(f(u)-f(u_h),\varphi-\mathrm{I}_h\varphi)\big\}+\big\{a(u-u_h,\mathrm{I}_h\varphi)\\ &\quad +(f(u)-f(u_h),\mathrm{I}_h\varphi)\big\}+\big\{(f(u_h)-f(u)+f'(u_h)(u-u_h),\varphi)\big\}\\ &= \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3. \end{aligned}\tag{4.24}$$

Using (4.23a) and the interpolation property, we can get

$$|\mathrm{I}_1|\le C(u,f)h\|u-u_h\|_1\|\varphi\|_2. \tag{4.25}$$

Using (4.23b) and the interpolation property, we can again get

$$|\mathrm{I}_3|\le C(u,f)\|(u-u_h)^2\|\|\varphi\|_1\le C(\|(u-\mathrm{I}_hu)^2\|+\|\theta^2\|)\|\varphi\|_1\le Ch^3\|\varphi\|_1. \tag{4.26}$$

Rewriting I_2 as

$$\begin{aligned}\mathrm{I}_2 &= a(u-u_h,\mathrm{I}_h\varphi)+(f(u)-f(u_h),\mathrm{I}_h\varphi)\\ &= a(u,\mathrm{I}_h\varphi)+(f(u),\mathrm{I}_h\varphi)-a(u_h,\mathrm{I}_h\varphi)-(f(u_h),\mathrm{I}_h\varphi)\\ &= (g,\mathrm{I}_h\varphi)-a(u_h,\mathrm{I}_h\varphi)-(f(u_h),\mathrm{I}_h\varphi)-(g,\mathrm{I}_h^*\varphi)+a(u_h,\mathrm{I}_h^*\varphi)-(\mathrm{I}_hf(u_h),\mathrm{I}_h^*\varphi)\\ &= (g,\mathrm{I}_h\varphi-\mathrm{I}_h^*\varphi)-a(u_h,\mathrm{I}_h\varphi-\mathrm{I}_h^*\varphi)-(f(u_h)-\mathrm{I}_hf(u_h),\mathrm{I}_h\varphi)-(\mathrm{I}_hf(u_h),\mathrm{I}_h\varphi-\mathrm{I}_h^*\varphi),\end{aligned}$$

and applying Lemma 3.5, Lemma 3.7 and Lemma 3.9, we get

$$|\mathrm{I}_2|\le C(h^3+h\|u-u_h\|_1)\|\varphi\|_1. \tag{4.27}$$

Therefore, substituting (4.25), (4.26), (4.27) and (4.21) into (4.24) yields

$$\|u-u_h\|^2\le|\mathrm{I}_1|+|\mathrm{I}_2|+|\mathrm{I}_3|\le C(h^3+h\|u-u_h\|_1)\|u-u_h\|.$$

Omitting the common factor $\|u-u_h\|$, this implies

$$\|u-u_h\|\le C(h^3+h\|u-u_h\|_1),$$

which gives the desired estimate (4.22) by using Theorem 4.1. □

Theorem 4.3. *Assume $f'(s)>0$, $f\in C^2(R)$, $g\in L^2(\Omega)$, and that the coefficients a_{12}, a_{21} in* (2.1) *satisfy $a_{12}=a_{21}$ and $\mathcal{J}_h$ is quasi-uniform triangular partition of domain Ω. Let $u\in H_0^1(\Omega)\cap C^(\Omega)$ be the solution of* (2.1) *and $u_h\in S_{0h}$ be the approximate solution of finite volume element method* (2.5) *with interpolated coefficients, respectively. Then*

$$\|u-u_h\|_{0,\infty}\le C(u,f,g)h^3|\ln h|. \tag{4.28}$$

Proof. By using the triangle inequality, we have

$$\|u-u_h\|_{0,\infty}\le\|u-\tilde{u}_h\|_{0,\infty}+\|\tilde{u}_h-u_h\|_{0,\infty},$$

where $\tilde{u}_h$ is the finite element approximation of u satisfying

$$a(\tilde{u}_h,v_h)+(f(\tilde{u}_h),v_h)=(g,v_h),\quad \forall v_h\in S_{0h}. \tag{4.29}$$

It has been shown in [7,9,23]

$$\|u-\tilde{u}_h\|_{0,\infty}\le C(u,f,g)h^3|\ln h|. \tag{4.30}$$

Next, we turn our attention to the estimate of $\|\tilde{u}_h-u_h\|_{0,\infty}$. Let $P^*\in K_0\subset\mathcal{J}_h$ such that $\|\tilde{u}_h-u_h\|_{0,\infty}=|(\tilde{u}_h-u_h)(P^*)|$ and $\delta_{P^*}\in C_0^\infty(\Omega)$ is a regularized Dirac δ-function satisfying

$$(\delta,v_h)=v_h(P^*).$$

Consider the corresponding regularized Green's function $G\in H_0^1(\Omega)$, defined by

$$a(G,v)+(f'(\tilde{u}_h)G,v)=(\delta_{P^*},v),\quad \forall v\in H_0^1(\Omega). \tag{4.31}$$

Let $G_h\in S_0^h$ be the finite element approximation of G, i.e.,

$$a(G-G_h,v)+(f'(\tilde{u}_h)(G-G_h),v)=0,\quad \forall v\in H_0^1(\Omega).$$

Then, in terms of (3.2) and (4.29), we can get

$$\begin{aligned}\|\tilde{u}_h-u_h\|_{0,\infty}=&(\delta_{P^*},\tilde{u}_h-u_h)=a(\tilde{u}_h-u_h,G_h)+(f'(\tilde{u}_h)(\tilde{u}_h-u_h),G_h)\\=&(g,G_h)-(f(\tilde{u}_h),G_h)-a(u_h,G_h)+(f'(\tilde{u}_h)(\tilde{u}_h-u_h),G_h)\\&+a(u_h,\mathrm{I}_h^*G)+(\mathrm{I}_hf(u_h),\mathrm{I}_h^*G_h)-(g,\mathrm{I}_h^*G_h)\\=&\big\{(g,G_h-\mathrm{I}_h^*G_h)-a(u_h,G_h-\mathrm{I}_h^*G_h)\big\}+\big\{(\mathrm{I}_hf(u_h),\mathrm{I}_h^*G_h)\\&-(f(u_h),G_h)\big\}+(f'(\tilde{u}_h)(\tilde{u}_h-u_h)-f(\tilde{u}_h)+f(u_h),G_h)\\=&\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6.\end{aligned} \tag{4.32}$$

Using Lemma 3.7, Lemma 3.9 and Theorem 4.1, we can get

$$|\mathrm{I}_4|\le Ch^3\|g\|_1\|G_h\|_1+C(h\|u-u_h\|_1+h^3\|u\|_2)\|G_h\|_1\le C(u,g)h^3\|G_h\|_1. \tag{4.33}$$

Using Lemma 3.7 and the interpolation property, we have

$$|\mathrm{I}_5|=|(f(u_h),G_h-\mathrm{I}_h^*G_h)|+|(f(u_h)-\mathrm{I}_hf(u_h),\mathrm{I}_h^*G_h)|\le C(u,f)h^3\|G_h\|_1. \tag{4.34}$$

Using (4.23b) and (4.34) and Theorem 4.1, we get

$$\begin{aligned}|\mathrm{I}_6|\le&|(f'(\tilde{u}_h)(\tilde{u}_h-u_h)-f(\tilde{u}_h)+f(u_h),G_h)|\\\le&C\|(\tilde{u}_h-u_h)^2\|\|G_h\|\le C_1h^3\|G_h\|.\end{aligned} \tag{4.35}$$

Table 1: Errors of FVEM with interpolated coefficients.

	H^1-seminorm		L^2-norm		L^∞-norm	
h	Error	Rate	Error	Rate	Error	Rate
1/4	$3.0209e-6$		$5.2118e-6$		$5.9708e-6$	
1/8	$9.5310e-7$	3.17	$7.2956e-7$	7.14	$7.7624e-7$	7.69
1/16	$2.4847e-7$	3.84	$8.6112e-8$	8.47	$9.9410e-8$	7.81
1/32	$6.2020e-8$	4.01	$9.7218e-9$	8.86	$1.1723e-8$	8.48

In addition in view of [7, 24] we get

$$\|G_h\|_1 \le C|\ln h|^{1/2}. \tag{4.36}$$

Combining of (4.33)-(4.35), we obtain

$$\|\tilde{u}_h - u_h\|_{0,\infty} \le Ch^3|\ln h|^{1/2}.$$

From this and (4.30) we get

$$\|u-u_h\|_{0,\infty} \le C(1+|\ln h|^{-1/2})h^3|\ln h|,$$

which gives the desired estimate (4.28) for sufficiently small h. $\square$

5 Numerical example

In this section we present a numerical experiment to verify the theoretical investigations. Let $\Omega=(0,1)\times(0,1)$. We choose $f(u)=u^3$ and $g(x,y)=2(x(1-x)+y(1-y))\cos(x(1-y))+y(1-x)(x^2+(1-y)^2)\sin(x(1-y))+y^3(1-x)^3\sin^3(x(1-y))$ in the problem (2.1) so that the exact solution is: $u(x,y)=y(1-x)\sin(x(1-y))$.

Place a right triangular decomposition on the domain $\Omega=(0,1)\times(0,1)$ with the right-angle-side length

$$h=\frac{1}{N}, \quad x_i=\frac{i}{N}, \quad y_j=\frac{j}{N}, \quad i,j=0,1,2,\cdots,N,$$

depicted as Fig. 4.

Compute it by the quadratic triangular finite volume element method with interpolated coefficients. The results are listed in Table 1. As observed, the error between the quadratic triangular finite volume element solution with interpolated coefficients and the exact solution is minor and stable at the nodes. The H^1-norm error is of the 2-order accuracy and the L^2-norm error and L^∞-norm error are of the 3-order accuracy. This agrees well with the theoretical analysis.

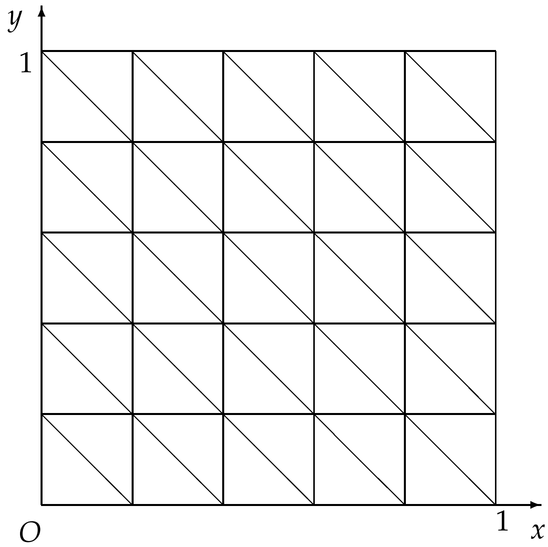

Figure 4: The right triangulation of $\Omega=(0,1)\times(0,1)$ with the right-angle-side length $h=1/5$.

Acknowledgments

This work is supported by National Natural Science Foundations of China under Project (No. 11571102). The authors would like to thank the reviewers for their suggestions to improve the quality of the paper.

References

[1] R. A. ADAMS, Sobolev Spaces, Pure and Applied Mathematics, Vol. 65, Academic Press (A subsidiary of Harcourt Brace Jovanovich, Publishers, New York, 1975.
[2] R. E. BANK AND D. J. ROSE, *Some error estimates for the box method,* SIAM J. Numer. Anal., 24 (1987), pp. 777–787.
[3] C. J. BI, *Superconvergence of finite volume element method for a nonlinear elliptic problem,* Numer. Methods PDEs, 23 (2007), pp. 220–233.
[4] J. H. BRAMBLE, *A second-order finite difference analog of the first biharmonia boundary value problem*, Numer. Math., 9(4) (1966), pp. 236–249.
[5] Z. Q. CAI, *On the finite volume element method,* Numer. Math., 58 (1991), pp. 713–735.
[6] Z. Q. CAI AND STEVE MCCORMICK, *On the accuracy of the finite volume element method for diffusion equations on composite grids,* SIAM J. Numer. Anal., 27 (1990), pp. 636–655.
[7] P. CHATZIPANTELIDIS, *Finite volume methods for elliptic PDE's: a new approach,* M2AN Math. Model. Numer. Anal., 3 (2002), pp. 307–324.
[8] P. CHATZIPANTELIDIS, V. GINTING AND R. D. LAZAROV, *A finite volume element for a nonlinear elliptic problem,* Numer. Linear Algebra Appl., 12 (2005), pp. 515–546.
[9] C. M. CHEN AND Y. Q. HUANG, High Accuracy Theory of Finite Element Methods, Hunan Science and Technology Publishers, 1995, in Chinese.

[10] C. M. CHEN, S. LARSON AND N. Y. ZHANG, *Error estimates of optimal order for finite element methods interpolated coefficients for the nonlinear heat equation*, IMA J. Numer. Anal., 9 (1989), pp. 507–524.

[11] S. H. CHOU, D. Y. KWAK AND Q. LI, *L^p error estimates and superconvergence for covolume or finite volume element methods,* Numer. Methods PDEs, 19 (2003), pp. 463–486.

[12] S. H. CHOU AND Q. LI, *Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: a unified approach,* Math. Comput., 69 (2000), pp. 103–120.

[13] JR J. DOUGLAS AND T. DUPONT, *A Galerkin method for a nonlinear Dirichlet problem*, Math. Comput., 29 (1975), pp. 689–696.

[14] R. E EWING, R. LAZAROV AND Y. LIN, *Finite volume element approximations of nonlocal reactive flows in porous media,* Numer. Methods PDEs, 16 (2000), pp. 285–311.

[15] R. E. EWING, T. LIN AND Y. P. LIN, *On the accuracy of the finite volume element method based on piecewise linear polynomials,* SIAM J. Numer. Anal., 39 (2002), pp. 1865–1888.

[16] J. FREHSE AND R. RANNACHER, *Asymptotic L_∞-error estimate for linear finite element appproximations of quasilinear boundary value problems,* SIAM J. Mumer. Anal., 15 (1978), pp. 418–431.

[17] J. G. HUANG AND L. K. LI, *Some superconvergence results for the covolume method for elliptic problems,* Commun. Numer. Meth. Eng., 17 (2001), pp. 291–302.

[18] J. G. HUANG AND S. T. XI, *On the finite volume element method for general self-adjoint elliptic problems,* SIAM J. Numer. Anal., 35 (1998), pp. 1762–1774.

[19] S. LARSON, V. THOMEE AND N. Y. ZHANG, *Interpolation of coefficients and transformation of dependent variable in element methods for the nonlinear heat equation,* Math. Methods Appl. Sci., 11 (1989), pp. 105–124.

[20] R. H. LI, *Generalized differential methods for a nonlinear Dirichlet problem,* SIAM J. Numer. Anal., 24 (1987), pp. 77–88.

[21] R. H. LI, Z. Y. CHEN AND W. WU, Generalized Difference Methods for Differential Equations, volume 226 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc, New York, 2000, Numerical Analysis of Finite Volume Methods.

[22] Y. H. LI AND R. H. LI, *Generalized diffierence methods on arbitrary quadrilateral networks,* J. Comput. Math., 17(6) (1999).

[23] Y. MATSUZAWA, *Finite element approximation for some quasilinear elliptic problems,* J. Comput. Appl. Math., 96 (1998), pp. 13–25.

[24] R. H. NOCHETTO, *Pointwise a posteriori error estimates for elliptic problems on highly graded meshes,* Math. Comput., 64 (1995), pp. 1–22.

[25] V. SELMIN, *The node-centred finite volume approach: Bridge between finite differences and finite elements*, Comput. Methods Appl. Mech. Eng., 102 (1993), pp. 107–138.

[26] E. SULI, *Convergence of finite volume schemes for Poissons equation on nonuniform meshes,* SIAM J. Numer. Anal., 28 (1991), pp. 1419–1430.

[27] E. SULI, *The accuracy of cell vertex finite volume methods on quadrilateral meshes*, Math. Comput., 59 (1992), pp. 359–382.

[28] H. J. WU AND R. H. LI, *Error estimates for finite volume element methods for general second-order elliptic problem,* Numer. Methods PDEs, 19 (2003), pp. 693–708.

[29] Z. G. XIONG AND C. M. CHEN, *Superconvergence of triangular quadratic finite element method with interpolated coefficients for nonlinear elliptic problem,* Acta Mathematics Scientia, Ser. A, 26(2) (2006), pp. 174–182.

[30] Z. G. XIONG AND C. M. CHEN, *Superconvergence of rectangular finite element with interpolated coefficients for semilinear elliptic problem,* Appl. Math. Comput., 181 (2006), pp. 1577–1584.

[31] Z. G. XIONG AND C. M. CHEN, *Superconvergence of continuous finite elements with interpo-*

lated coefficients for initial value problems of nonlinear ordinary differential equation, Numerical Mathematics: A Journal of Chinese Universities, (English Ser.), 16(1) (2007), pp. 37–44.

[32] Z. G. XIONG AND Y. P. CHEN, *Finite volume element method with interpolated coefficients for two-point boundary value problem of semilinear differential equations,* Comput. Methods Appl. Mech. Eng., 196 (2007), pp. 3798–3804.

[33] Z. G. XIONG AND Y. P. CHEN, *Triangular finite volume element method with interpolated coefficients for a semilinear elliptic equation,* J. Comput. Math., 32(2) (2014), pp. 152–168.

[34] P. ZHU AND R. LI, *Generalized difference methods for second order elliptic partial differential equations (II)-quadrilateral grids,* Numerical Mathematics: A Journal of Chinese Universities, 4 (1982), pp. 360–375.

[35] M. ZLAMAL, *A finite element solution of the nonlinear heat equation,* RAIRO Model. Anal. Numer., 14 (1980), pp. 203–216.