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Nonconforming Finite Element Method for the Transmission Eigenvalue Problem

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Abstract. In this paper, we analyze a nonconforming finite element method for the computation of transmission eigenvalues and the corresponding eigenfunctions. The error estimates of the eigenvalue and eigenfunction approximation are given, respectively. Finally, some numerical examples are provided to validate the theoretical results.

AMS subject classifications: 34L16, 65M60

Key words: Transmission eigenvalue, Morley element, nonconforming finite element method.

1 Introduction

The transmission eigenvalue problem arises in the study of the inverse scattering by inhomogeneous media. Due to the important applications in the inverse scattering theory, the transmission eigenvalue problem attracted more and more attention recently [5–7,10–12,17,20]. It not only has the theoretical importance [10], but also can be used to recover the properties of the scattering material [4,6,23] since they can be determined from the scattering data.

In the past few years, the existence theory and application of transmission eigenvalue have been developed, some details can be found in the recent survey paper by Cakoni and Haddar [7]. However, in contrast, the numerical treatment of transmission eigenvalues and the associated interior transmission problem is very limited [1,11,14–16,18,19,24,25]. To the best of the authors' knowledge, the recent paper by Colton, Monk, and Sun [11] contains the first numerical study where three finite element methods are proposed. Sun [24] proposes two iterative methods (bisection and secant). Ji, Sun and Turner [15] construct a mixed finite element method. The technique is employed in [19]

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to compute the Maxwell's transmission eigenvalues. Most papers do not discuss the convergence due to the difficulty that the problem is neither elliptic nor self-adjoint. Some error estimates for the eigenvalues are provided in [8, 24].

In [16], the convergence analysis of the conforming finite element method and the corresponding multigrid method have been given for the transmission eigenvalue problem. The aim of this paper is to give the convergence analysis for the transmission eigenvalue problem by the nonconforming finite element method.

The rest of this paper is organized as follows. In Section 2, we introduce the transmission eigenvalue problem and derive an equivalent fourth order reformulation. The nonconforming finite element method and its error estimates are given in Section 3. In Section 4, three examples are presented to validate the derivative theoretical results. The last section gives some concluding remarks.

2 Transmission eigenvalue problem

First, we will introduce some notations. Symbols $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$ mean $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$, respectively, where C_1, c_2, c_3 and C_3 are constants independent of the mesh size. C (with or without subscript, uppercase or lowercase) denotes a generic positive constant which may take different value at its different occurrences through the paper.

From the physical standpoint, in this paper, we only study the real transmission eigenvalues corresponding to the scattering of acoustic waves by a bounded simply connected inhomogeneous medium $\Omega \subset \mathcal{R}^2$. The transmission eigenvalue problem is to find $k \in \mathcal{R}$, $\phi, \varphi \in H^2(\Omega)$, $\phi - \varphi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta\phi + k^2 n(x)\phi = 0 & \text{in } \Omega, \\ \Delta\varphi + k^2\varphi = 0 & \text{in } \Omega, \\ \phi - \varphi = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial\phi}{\partial\nu} - \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ν is the unit outward normal to $\partial\Omega$. The index of refraction $n(x)$ satisfies $n(x) > \alpha_0$ a.e. in Ω for some constant $\alpha_0 > 1$ or $0 < n(x) < \tilde{\alpha}_0$ a.e. in Ω for some constant $\tilde{\alpha}_0 < 1$. We call k the transmission eigenvalues if it makes (2.1) has a nontrivial solution.

In order to simplify the notation, we define

$$V := H_0^2(\Omega) = \Big\{ u \in H^2(\Omega) : \ u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \Big\}, \tag{2.2}$$

and denote (u, v) the standard $L^2(\Omega)$ inner product.

Let $u = \phi - \varphi \in V$. Then we have

$$(\Delta + k^2)u = -k^2\big(n(x) - 1\big)\phi. \tag{2.3}$$

Thus $(n(x)-1)^{-1}(\Delta+k^2)u=-k^2\phi$. We apply $(\Delta+k^2n(x))$ to both sides of this equation to obtain a fourth order problem

$$\big(\Delta+k^2n(x)\big)\frac{1}{n(x)-1}(\Delta+k^2)u=0. \tag{2.4}$$

The transmission eigenproblem can be stated as: Find $(k^2\neq 0,u)\in\mathcal{R}\times V$ such that

$$\Big(\frac{1}{n(x)-1}(\Delta u+k^2u),\Delta v+k^2n(x)v\Big)=0,\quad \forall v\in V. \tag{2.5}$$

Next, we use the idea in [24] to introduce the associated generalized eigenvalue problem. Let $\tau=k^2$ (we also call τ a transmission eigenvalue if k is) and define

$$\mathcal{A}_\tau(u,v)=\Big(\frac{1}{n(x)-1}(\Delta u+\tau u),(\Delta v+\tau v)\Big)+\tau^2(u,v), \tag{2.6}$$

for $n(x)\geq\alpha_0$ a.e. in Ω for some constant $\alpha_0>1$, and

$$\begin{aligned}\widetilde{\mathcal{A}}_\tau(u,v)&=\Big(\frac{1}{1-n(x)}(\Delta u+\tau n(x)u),(\Delta v+\tau n(x)v)\Big)+\tau^2\big(n(x)u,v\big)\\&=\Big(\frac{n(x)}{1-n(x)}(\Delta u+\tau u),(\Delta v+\tau v)\Big)+\big(\Delta u,\Delta v\big),\end{aligned} \tag{2.7}$$

for $0<n(x)\leq\tilde{\alpha}_0$ a.e. in Ω for some constant $\tilde{\alpha}_0<1$. We also define

$$\mathcal{B}(u,v)=\big(\nabla u,\nabla v\big). \tag{2.8}$$

From (2.5) and (2.6)-(2.7), it is obvious to show that the transmission eigenvalue problem can be written as: Find $(\tau,u)\in\mathcal{R}\times V$ such that $\mathcal{B}(u,u)=1$ and

$$\mathcal{A}_\tau(u,v)=\tau\mathcal{B}(u,v),\quad \forall v\in V, \tag{2.9}$$

when $n(x)\geq\alpha_0$ a.e. in Ω for some constant $\alpha_0>1$, and

$$\tilde{\mathcal{A}}_\tau(u,v)=\tau\mathcal{B}(u,v),\quad \forall v\in V, \tag{2.10}$$

when $0<n(x)\leq\tilde{\alpha}_0$ a.e. in Ω for some constant $\tilde{\alpha}_0<1$.

The bilinear forms $\mathcal{A}_\tau(\cdot,\cdot)$, $\tilde{\mathcal{A}}_\tau(\cdot,\cdot)$ and $\mathcal{B}_\tau(\cdot,\cdot)$ have the following properties.

Lemma 2.1 (see [7,24])**.** *The bilinear form $\mathcal{B}(\cdot,\cdot)$ is a symmetric and nonnegative bilinear form on $V\times V$. If*

$$\frac{1}{n(x)-1}\geq\gamma$$

a.e. in Ω for some $\gamma>0$, the bilinear form $\mathcal{A}_\tau(\cdot,\cdot)$ is coercive on $V\times V$ and if

$$\frac{n(x)}{1-n(x)}\geq\tilde{\gamma}$$

a.e. in Ω for some $\tilde{\gamma}>0$, the bilinear form $\tilde{\mathcal{A}}_\tau(\cdot,\cdot)$ is coercive on $V\times V$.

Here, we use the idea in [24] to transform the transmission eigenvalue as a fix point of a nonlinear function. For this aim, we define the following generalized eigenvalue problem: Find $(\lambda(\tau),u)\in\mathcal{R}\times V$ such that $\mathcal{B}(u,u)=1$ and

$$\mathcal{A}_\tau(u,v)=\lambda(\tau)\mathcal{B}(u,v),\quad \forall v\in V, \tag{2.11}$$

for $n(x)\geq\alpha_0$ a.e. in Ω for some constant $\alpha_0>1$, or

$$\tilde{\mathcal{A}}_\tau(u,v)=\lambda(\tau)\mathcal{B}(u,v),\quad \forall v\in V, \tag{2.12}$$

for $0<n(x)\leq\tilde{\alpha}_0$ a.e. in Ω for some constant $\tilde{\alpha}_0<1$.

Then $\lambda(\tau)$ is a nonlinear function of τ. Furthermore, from the definitions of $A_\tau(\cdot,\cdot)$ and $\tilde{A}_\tau(\cdot,\cdot)$, $\lambda(\tau)$ is continuous corresponding to τ based on the eigenvalue perturbation theory (c.f. [2, 3]). From (2.9) and (2.10), a transmission eigenvalue is a root of the following nonlinear function

$$f(\tau):=\lambda(\tau)-\tau. \tag{2.13}$$

This paper considers the numerical method for (2.13). We refer the readers to [7] about the existence result.

3 Nonconforming finite element method

In this section, we first introduce the nonconforming finite element method for the transmission eigenvalue problem. Then the results in [24] with the bisection way and the method in [16] are adopted to give the error estimates of the eigenvalue approximation. Finally, based on the theory of the nonconforming finite element method for the eigenvalue problem, the error estimates for the eigenfunction approximation are also derived.

3.1 Error estimate of the eigenvalue approximation

For simplicity, we are only concerned with the case (2.9), (2.10) follows similarly. We denote $M(\lambda(\tau))$ the following generalized eigenfunction set corresponding to the eigenvalue $\lambda(\tau)$

$$M(\lambda(\tau))=\big\{v\in V:\ v \text{ is a generalized eigenfunction of } (2.11) \text{ corresponding to } \lambda(\tau) \text{ and } \mathcal{B}(v,v)=1\big\}.$$

We also need the operator $T_\tau:H^1(\Omega)\to V$

$$\mathcal{A}_\tau(T_\tau f,v)=\mathcal{B}(f,v),\quad \forall f\in H^1(\Omega), \tag{3.1}$$

which is compact and self-adjoint because of the compact embedding of V into $H^1(\Omega)$. We rewite the eigenvalue problem (2.11) as

$$\lambda(\tau)T_\tau u_\tau=u_\tau. \tag{3.2}$$

So $\mu = 1/\lambda(\tau)$ is an eigenvalue of T_τ with the eigenfunction u_τ.

We assume $\mathcal{T}_h$ a shape regular mesh over Ω with mesh size h. Based on the mesh $\mathcal{T}_h$, we define a nonconforming finite element space V_h (for example, the Morley element discussed in this paper) such that $V_h \not\subset V$. The degrees of freedom of the Morley element are the function values at the vertices of the mesh $\mathcal{T}_h$ and the normal derivatives at the midpoints on the edges of $\mathcal{T}_h$. The nonconforming finite element space of Morley can be defined as follows:

$$\begin{aligned} V_h = \big\{ v \in L^2(\Omega):\ & v|_K \in \mathcal{P}_2,\ v \text{ and } \partial_\nu v \text{ are continuous at vertices and} \\ & \text{the midpoints of edges, and vanish on the boundary } \partial\Omega,\ \forall K \in \mathcal{T}_h \big\}, \end{aligned}$$

where $\mathcal{P}_2$ denotes the polynomial space of degree less than or equal to two.

In order to analyze the error estimate of the eigenfunction approximation by the nonconforming finite element method, we define the following norm

$$\|v\|_h := \Big(\sum_{K\in\mathcal{T}_h} \|\nabla^2 v\|_{0,K}^2 \Big)^{\frac{1}{2}}.$$

It is well known that the $\|\cdot\|_h$ is a norm in the space $V+V_h$ and $\|v\|_h = \|v\|_V$ when $v\in V$ (cf. [21]).

Now we can define the following discrete version of the eigenvalue problem (2.9) as: Find $(\tau_h, u_h) \in \mathcal{R}\times V_h$ such that $\mathcal{B}(u_h,u_h)=1$ and

$$\mathcal{A}_{\tau_h,h}(u_h,v_h) = \tau_h \mathcal{B}_h(u_h,v_h), \quad \forall v_h \in V_h, \tag{3.3}$$

where we assume $n(x)$ is a constant function and

$$\mathcal{A}_{\tau_h,h}(u_h,v_h) := \sum_{K\in\mathcal{T}_h} \int_K \frac{1}{n(x)-1} (\nabla^2 u_h \nabla^2 v_h - 2\tau_h \nabla u_h \nabla v_h) dK + \tau_h^2 \Big(\frac{n(x)}{n(x)-1} u_h, v_h \Big), \tag{3.4a}$$

$$\mathcal{B}_h(u_h,v_h) = \sum_{K\in\mathcal{T}_h} \int_K \nabla u_h \nabla v_h dK. \tag{3.4b}$$

To analyze the error estimates by the nonconforming finite element method, similarly to [16], we follow the same idea in [24] to find the roots of a discrete version of (2.13): Find $(\lambda_h(\tau), \hat{u}_h) \in \mathcal{R}\times V_h$ such that $\mathcal{B}_h(\hat{u}_h,\hat{u}_h)=1$ and

$$\mathcal{A}_{\tau,h}(\hat{u}_h,v_h) = \lambda_h(\tau) \mathcal{B}_h(\hat{u}_h,v_h), \quad \forall v_h \in V_h. \tag{3.5}$$

We also need the discrete operator $T_{\tau,h}: H^1(\Omega) \to V_h$

$$\mathcal{A}_{\tau,h}(T_{\tau,h} f, v) = \mathcal{B}_h(f,v), \quad \forall v \in V_h, \tag{3.6}$$

which is a compact self-adjoint operator due to the fact that any discrete operator is compact [22] and (3.5) can be rewritten as

$$\lambda_h(\tau) T_{\tau,h} \hat{u}_h = \hat{u}_h. \tag{3.7}$$

Since T_τ relates to a lower order perturbed biharmonic operator, the eigenfunction in $M(\lambda)$ satisfies

$$\|u\|_{2+\gamma}<\infty \quad \text{for } \ u\in M(\lambda), \tag{3.8}$$

here γ $(0<\gamma\leq 1)$ is the regular parameter which depends on the maximal interior angle of the boundary $\partial\Omega$ and $\gamma=1$ for the convex domain [13].

We have the following error estimate by the standard theory of the nonconforming finite element method for the eigenvalue problem (c.f. [2,3,21]).

Lemma 3.1. *Assume $n(x)$ satisfies the conditions of Lemma 2.1 and $(\lambda_h(\tau),\hat{u}_h)$ is a solution to* (3.5). *Then there exists a solution of the exact eigenvalue problem* (2.11) *satisfying the following error estimate*

$$|\lambda_h(\tau)-\lambda(\tau)|\lesssim h^{2\gamma}. \tag{3.9}$$

The eigenvalue τ_h in (3.3) is a root of the following equation

$$f_h(\tau):=\lambda_h(\tau)-\tau. \tag{3.10}$$

The following result shows that the roots of (3.10) approach the roots of (2.13) well if the mesh is fine enough.

Theorem 3.1. *Under the conditions of Lemma 3.2 in [24], assume τ_h is a solution of* (3.3) *approximating the exact eigenvalue τ of* (2.11). *Then for small enough h, we have the following error estimate*

$$|\tau-\tau_h|\lesssim h^{2\gamma}. \tag{3.11}$$

Proof. Based on Lemma 3.2 in [24], we have $f_h'(\tau)<-C<0$ with $C>0$ for small enough h. From Lemma 3.1, we have

$$|f(\tau)-f_h(\tau)|\lesssim h^{2\gamma},$$

on an interval $[a-h^{2\gamma}/C,b+h^{2\gamma}/C]$ with $0<a<b$. Then from Lemma 3.3 in [24] with $\lambda(\tau)=\tau$, (3.11) is obtained. □

3.2 Error estimate of the eigenfunction approximation

In this subsection, we will give the convergence analysis of the eigenfunction approximation. Here, we set τ_h to be the approximation of the exact eigenvalue τ defined by (2.9), i.e., $\lambda(\tau)=\tau$.

An auxiliary eigenvalue problem is constructed first: Find $(\tilde{\lambda},\tilde{u})\in\mathcal{R}\times V$ such that $\mathcal{B}(\tilde{u},\tilde{u})=1$ and

$$\mathcal{A}_{\tau_h}(\tilde{u},v)=\tilde{\lambda}\mathcal{B}(\tilde{u},v), \quad \forall v\in V. \tag{3.12}$$

Correspondingly, a solution operator $T_{\tau_h}: H^1(\Omega) \to V$ can be defined as

$$\mathcal{A}_{\tau_h}(T_{\tau_h} f, v) = \mathcal{B}(f, v), \quad \forall v \in V. \tag{3.13}$$

Then (3.12) can be rewritten as

$$\tilde{\lambda} T_{\tau_h} \tilde{u} = \tilde{u}. \tag{3.14}$$

From Theorem 3.1, we know $\tau_h \to \tau$ and $T_{\tau_h} \to T_\tau$ in the operator norm as $h \to 0$.

Lemma 3.2. *Assume eigenvalues τ and τ_h have the error estimate* (3.11)*. The two operators T_τ and T_{τ_h} have the following estimate*

$$\|T_\tau - T_{\tau_h}\|_V \lesssim |\tau - \tau_h| \lesssim h^{2\gamma}, \tag{3.15}$$

where $\|\cdot\|_V$ means the operator norm on $V \to V$.

Proof. The proof uses the same idea of Lemma 3.3 in [16]. □

To estimate $\|u - \tilde{u}\|_V$, we denote by Γ a circle in the complex plane centered at $1/\tau$ such that no other eigenvalues of T_τ lie inside and define $R_z(T_\tau) = (z - T_\tau)^{-1}$, $R_z(T_{\tau_h}) = (z - T_{\tau_h})^{-1}$. The spectral projection operator is defined as

$$E_\tau = \frac{1}{2\pi \mathrm{i}} \int_\Gamma R_z(T_\tau) dz. \tag{3.16}$$

The operator E_τ is a projection onto the space of generalized eigenvectors associated with $1/\tau$ and T_τ, i.e., $R(E_\tau) = N((1/\tau - T_\tau)^\alpha)$, where R denotes the range and α is the ascent of τ. For h sufficiently small, the spectral projection

$$E_{\tau_h} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma R_z(T_{\tau_h}) dz, \tag{3.17}$$

exists, E_{τ_h} is the spectral projection [3] associated with T_{τ_h} and the eigenvalues of T_{τ_h} inside Γ, $E_{\tau_h} \to E_\tau$ in norm and $\dim R(E_{\tau_h}) = \dim R(E_\tau)$.

Assume M and N are two subsets of V. We define the distance between M and N as

$$\delta(M,N) = \max(\hat{\delta}(M,N), \hat{\delta}(N,M)), \quad \text{where } \hat{\delta}(M,N) = \sup_{v \in M,\ \|v\|_h = 1} \inf_{\chi \in N} \|v - \chi\|_h.$$

Lemma 3.3. *The two finite dimensional spaces $R(E_\tau)$ and $R(E_{\tau_h})$ satisfy*

$$\delta(R(E_\tau), R(E_{\tau_h})) \lesssim \|(T_\tau - T_{\tau_h})|_{R(E_\tau)}\|_V, \tag{3.18}$$

for small enough h, where $(T_\tau - T_{\tau_h})|_{R(E_\tau)}$ means the restriction of $T_\tau - T_{\tau_h}$ onto $R(E_\tau)$.

Proof. The proof is similar to Theorem 7.1 in [3] and Lemma 3.4 in [16], we ignore the detail here. □

We can get the following result by the standard theory of operator perturbation.

Lemma 3.4. *Assume eigenvalues* τ *and* τ_h *have the error estimate* (3.11). *The eigenpair approximation* (τ,u) *of* (2.9) *and* $(\tilde{\lambda},\tilde{u})$ *of* (3.12) *satisfy*

$$\|u-\tilde{u}\|_V \lesssim h^{2\gamma}, \tag{3.19a}$$
$$|\tau-\tilde{\lambda}| \lesssim h^{2\gamma}. \tag{3.19b}$$

Proof. We can get the following estimate by Lemmas 3.2 and 3.3

$$\delta(R(E_\tau),R(E_{\tau_h})) \lesssim h^{2\gamma}. \tag{3.20}$$

Combining this estimate with the theory in [3], the desired estimates (3.19a) and (3.19b) can be obtained. □

Based on (3.19a)-(3.19b) and the triangle inequality, we can get the following error estimates of the eigenpair approximation (τ_h,u_h).

Theorem 3.2. *Under the conditions of Lemma 3.2 in [24], assume* τ_h *is a solution of* (3.3) *by the Morley element approximating the exact eigenvalue* τ *of* (2.9). *Then for small enough* h, *we have the following error estimates*

$$\|u_h-u\|_h \lesssim h^{\gamma}, \tag{3.21a}$$
$$\|u_h-u\|_1 \lesssim h^{2\gamma}, \tag{3.21b}$$
$$|\tau-\tau_h| \lesssim h^{2\gamma}. \tag{3.21c}$$

Proof. From Theorem 3.1, we have the error estimate (3.21c). Actually, the discrete eigenvalue problem (3.3) is the discretization of the linear eigenvalue problem (3.12) by the nonconforming finite element method.

By the standard theory [3,21], we can get the following estimates

$$\|\tilde{u}-u_h\|_h \lesssim h^{\gamma}, \tag{3.22a}$$
$$\|\tilde{u}-u_h\|_1 \lesssim h^{2\gamma}, \tag{3.22b}$$
$$|\tilde{\lambda}-\tau_h| \lesssim h^{2\gamma}. \tag{3.22c}$$

From Lemma 3.4, the following estimates hold

$$\|u-\tilde{u}\|_V \lesssim h^{2\gamma}, \tag{3.23a}$$
$$|\tau-\tilde{\lambda}| \lesssim h^{2\gamma}. \tag{3.23b}$$

The combination of (3.22a)-(3.22c) and (3.23a)-(3.23b) leads to the desired results (3.21a)-(3.21b) and the proof is completed. □

4 Numerical results

In this section, we solve the eigenvalue problem (2.9) on the unit square $\Omega=(0,1)\times(0,1)$ and the L-shape domain $\Omega=(-1,1)\times(-1,1)\backslash[0,1)\times(-1,0]$ by the Morley element method, respectively. For the unit square, as predicted in Theorem 3.2, the convergence order of the eigenvalue approximation is two, i.e.,

$$|\tau-\tau_h|\lesssim h^2, \tag{4.1}$$

since $\gamma=1$ in (3.8).

In order to check the convergence behavior of the nonconforming finite element method, we produce a sequence of finite element spaces which are constructed by using the Morley element on triangular mesh. In all the examples, we use triangular mesh with $h=1/4$ as the initial meshes and then we refine this mesh with the regular way (connecting the midpoints of each edge) to investigate the convergence behavior. The fines mesh is $h=1/64$.

4.1 Unit square with $n=16$

First, we give the numerical results of the Morley element on the unit square with the index of refraction $n=16$.

The first four eigenvalues ($k_h=\sqrt{\tau_h}$) on the finest mesh are (1.879314, 2.443571, 2.443572, 2.865203) which are consistent with the results in [15,16]. Fig. 1 presents the error estimates of the numerical approximation for the first four eigenvalues. From Fig. 1, we know that the Morley element can obtain the theoretically predicted second convergence order.

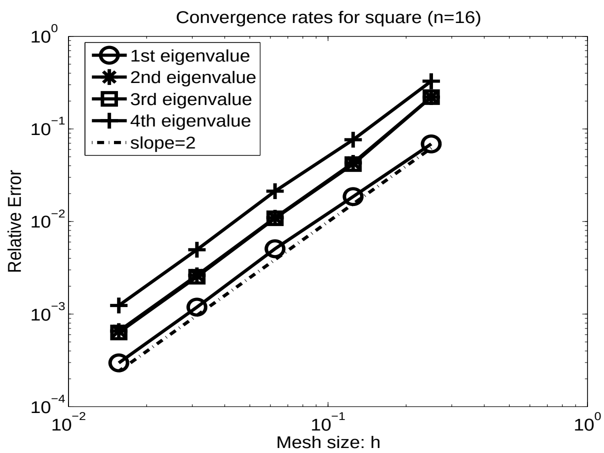

Figure 1: Error estimates for unit square with $n=16$.

4.2 Unit square with $n=8+x_1-x_2$

In the second example, we also consider the eigenvalue problem on the unit square $\Omega=(0,1)\times(0,1)$. Here, the index of refraction is chosen to be a function $n(x)=8+x_1-x_2$.

The first four eigenvalue approximations ($k_h=\sqrt{\tau_h}$) on the finest mesh are (2.823589, 3.538882, 3.539337, 4.117087) and Fig. 2 gives the numerical errors for the first four eigenvalues. Fig. 2 also shows that the Morley element method does obtain the theoretically predicted second convergence order.

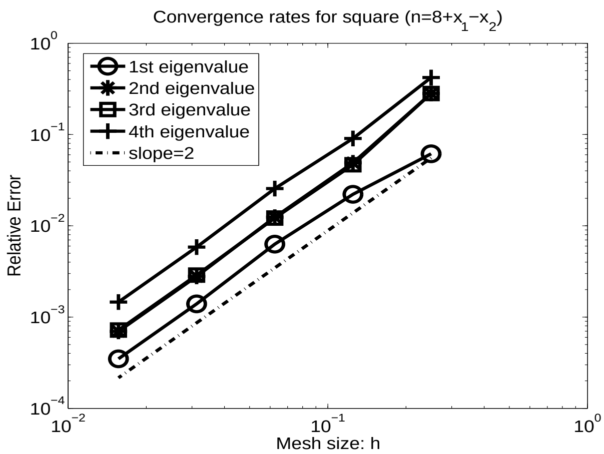

Figure 2: Error estimates for unit square with $n=8+x_1-x_2$.

4.3 L-shape domain with $n=16$

Here we study the Morley element on the L-shape domain with $n=16$. Since Ω has a reentrant corner, eigenfunctions with singularities are expected. The convergence order for the eigenvalue approximation may be less than 2 by the Morley element which is the order predicted by the theory for regular eigenfunctions.

We still give the first four eigenvalues on the finest mesh: (1.474002, 1.569339, 1.704406, 1.782807). Fig. 3 gives the numerical errors for the first four eigenvalues. As we have expected, the convergence order shown in Fig. 3 is only 1.5 for the first eigenvalue which is less than 2. Similarly to the elliptic eigenvalue problem, the second, third and fourth eigenvalues have better accuracy than the first one.

5 Concluding remarks

In this paper, we use the Morley element method to solve the transmission eigenvalue problem. The corresponding convergence analysis is given for the eigenvalue and eigenfunction approximation. Finally, three numerical examples are presented to confirm the

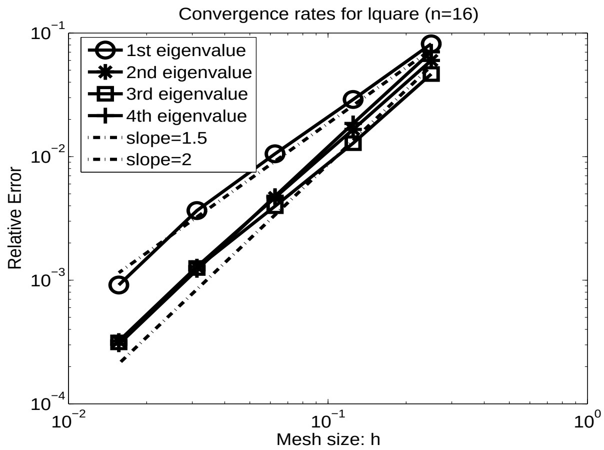

Figure 3: Error estimates for L-shape domain with $n=16$.

theoretical convergence order. The analysis used in this paper can also be extended to other nonconforming finite element methods for the transmission eigenvalue problem (e.g., [9,26]).

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