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Nonconforming Finite Element Method for the Transmission Eigenvalue Problem

Xia Ji, Yingxia Xi and Hehu Xie*

LSEC, NCMIS, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

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Abstract. In this paper, we analyze a nonconforming finite element method for the computation of transmission eigenvalues and the corresponding eigenfunctions. The error estimates of the eigenvalue and eigenfunction approximation are given, respectively. Finally, some numerical examples are provided to validate the theoretical results.

AMS subject classifications: 34L16, 65M60

Key words: Transmission eigenvalue, Morley element, nonconforming finite element method.

1 Introduction

The transmission eigenvalue problem arises in the study of the inverse scattering by inhomogeneous media. Due to the important applications in the inverse scattering theory, the transmission eigenvalue problem attracted more and more attention recently [5–7,10–12,17,20]. It not only has the theoretical importance [10], but also can be used to recover the properties of the scattering material [4,6,23] since they can be determined from the scattering data.

In the past few years, the existence theory and application of transmission eigenvalue have been developed, some details can be found in the recent survey paper by Cakoni and Haddar [7]. However, in contrast, the numerical treatment of transmission eigenvalues and the associated interior transmission problem is very limited [1,11,14–16,18, 19,24,25]. To the best of the authors' knowledge, the recent paper by Colton, Monk, and Sun [11] contains the first numerical study where three finite element methods are proposed. Sun [24] proposes two iterative methods (bisection and secant). Ji, Sun and Turner [15] construct a mixed finite element method. The technique is employed in [19]

*Corresponding author.
Email: jixia@lsec.cc.ac.cn (X. Ji), yxiaxi@lsec.cc.ac.cn (Y. Xi), hhxie@lsec.cc.ac.cn (H. Xie)

to compute the Maxwell's transmission eigenvalues. Most papers do not discuss the convergence due to the difficulty that the problem is neither elliptic nor self-adjoint. Some error estimates for the eigenvalues are provided in [8, 24].

In [16], the convergence analysis of the conforming finite element method and the corresponding multigrid method have been given for the transmission eigenvalue problem. The aim of this paper is to give the convergence analysis for the transmission eigenvalue problem by the nonconforming finite element method.

The rest of this paper is organized as follows. In Section 2, we introduce the transmission eigenvalue problem and derive an equivalent fourth order reformulation. The nonconforming finite element method and its error estimates are given in Section 3. In Section 4, three examples are presented to validate the derivative theoretical results. The last section gives some concluding remarks.

2 Transmission eigenvalue problem

First, we will introduce some notations. Symbols $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$ mean $x_1 \le C_1 y_1$, $x_2 \ge c_2 y_2$ and $c_3 x_3 \le y_3 \le C_3 x_3$, respectively, where C_1, c_2, c_3 and C_3 are constants independent of the mesh size. C (with or without subscript, uppercase or lowercase) denotes a generic positive constant which may take different value at its different occurrences through the paper.

From the physical standpoint, in this paper, we only study the real transmission eigenvalues corresponding to the scattering of acoustic waves by a bounded simply connected inhomogeneous medium $\Omega \subset \mathcal{R}^2$. The transmission eigenvalue problem is to find $k \in \mathcal{R}$, $\phi, \varphi \in H^2(\Omega)$, $\phi - \varphi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta\phi + k^2 n(x)\phi = 0 & \text{in } \Omega, \\ \Delta\varphi + k^2\varphi = 0 & \text{in } \Omega, \\ \phi - \varphi = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial\phi}{\partial\nu} - \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ν is the unit outward normal to $\partial\Omega$. The index of refraction $n(x)$ satisfies $n(x) > \alpha_0$ a.e. in Ω for some constant $\alpha_0 > 1$ or $0 < n(x) < \tilde{\alpha}_0$ a.e. in Ω for some constant $\tilde{\alpha}_0 < 1$. We call k the transmission eigenvalues if it makes (2.1) has a nontrivial solution.

In order to simplify the notation, we define

$$V := H_0^2(\Omega) = \Big\{ u \in H^2(\Omega) : \ u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \Big\}, \tag{2.2}$$

and denote (u,v) the standard $L^2(\Omega)$ inner product.

Let $u = \phi - \varphi \in V$. Then we have

$$(\Delta + k^2) u = -k^2 \big(n(x) - 1 \big) \phi. \tag{2.3}$$