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A Residual Distribution Method Using Discontinuous Elements for the Computation of Possibly Non Smooth Flows

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Abstract. In this paper, we describe a residual distribution (RD) method where, contrarily to "standard" this type schemes, the mesh is not necessarily conformal. It also allows to use discontinuous elements, contrarily to the "standard" case where continuous elements are requested. Moreover, if continuity is forced, the scheme is similar to the standard RD case. Hence, the situation becomes comparable with the Discontinuous Galerkin (DG) method, but it is simpler to implement than DG and has guaranteed L^∞ bounds. We focus on the second-order case, but the method can be easily generalized to higher degree polynomials.

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1 Introduction

This paper is devoted to the design of an approximation method for steady hyperbolic problems by means of a scheme which enjoys the most possible compact stencil. There exist many similar methods, for example the Discontinuous Galerkin method, or the continuous Residual Distribution schemes. In the first case, the solution is represented in each element of the mesh by polynomial functions where no continuity is enforced at the element boundaries. Hence, the method is very flexible since the mesh does not need to be conformal, nor the polynomial degree be the same in each element. Other approximation techniques than local polynomial representations can

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be chosen In our opinion, one of its disadvantages is its complexity, especially when one considers mixed hyperbolic/elliptic problems such as the Navier Stokes equations. Moreover, and this is the point we are interested in here, when discontinuous solutions are computed, the non linear non oscillatory stabilization mechanisms are not completely satisfactory because they depend on parameters or are quite complex to design, see [1–4] for example. Either they are very complex to set up, or they introduce too much dissipation.

In the case of the residual distribution (RD) methods, the solution is also approximated by piecewise polynomial functions, but here the approximation is globally continuous. Hence, the algorithmic complexity is lower (in term of memory especially). Another property is that there exists a very general and systematic method that enables us to guaranty accuracy formal $O(h^{k+1})$ accuracy, even at local extrema, and L^∞ stability. However, the mesh must be conformal, see [5–7] among several others.

In this note, we describe a residual distribution method where the functional representation does not need to be continuous across edges. The method is general and could be extended to any order of accuracy, following the lines of [8], but here, we have only developed it for a local P^1 interpolation in each element to present the ideas. Contrary to the "classical" RD schemes, the continuity across edges is no longer enforced. This method is simpler than the one described in [9]. Indeed, the scheme reduces to the one of [5, 10] and [6] if continuity is enforced across edges. Compared with standard DG methods, the scheme non oscillatory properties are obtained *without* any parameter.

The paper is organized as follows. We first describe the method for a scalar problem. Then the method is extended to the Euler equations for fluid dynamics. The extension to 3D is straightforward as well as on non conformal meshes. This paper opens the road for *h*-*p* adaptation for RD schemes.

This paper is a translation of a 2007 report written in French, [11], with some improvements. In the meantime, Hubbard [12] has published a similar technique. However, the similarity starts and ends in that we both use discontinuous elements. Hubbard then develops his method using an extension of the N scheme. We have used Lax-Friedrichs method, but following [13], any standard finite volume scheme can be rewritten as a RD scheme, and hence can be plugged into our framework. The method is also much simpler than the one in [9].

2 The scalar case

Let us consider the following problem, defined in $\Omega\subset\mathbb{R}^2$ to make the presentation simpler

$$\operatorname{div} f(u) = 0, \qquad \text{if } x \in \Omega, \tag{2.1a}$$

$$u = g, \qquad \text{if } x \in \Gamma^-, \tag{2.1b}$$