Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **2**, No. 1, pp. 56-75

DOI: 10.4208/aamm.09-m0931
February 2010

Superconvergence of Rectangular Mixed Finite Element Methods for Constrained Optimal Control Problem

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Received 10 April 2009; Accepted (in revised version) 03 July 2009
Available online 31 December 2009

Abstract. We investigate the superconvergence properties of the constrained quadratic elliptic optimal control problem which is solved by using rectangular mixed finite element methods. We use the lowest order Raviart-Thomas mixed finite element spaces to approximate the state and co-state variables and use piecewise constant functions to approximate the control variable. We obtain the superconvergence of $\mathcal{O}(h^{1+s})$ $(0<s\leq 1)$ for the control variable. Finally, we present two numerical examples to confirm our superconvergence results.

AMS subject classifications: 49J20, 65N30

Key words: Constrained optimal control problem; linear elliptic equation; mixed finite element methods; rectangular partition; superconvergence properties.

1 Introduction

In this paper, we focus on the superconvergence properties of rectangular mixed finite element methods for linear elliptic optimal control problem. Optimal control problems are playing increasingly important role in the design of modern life. They have various applications in the operation of physical, social, and economic processes. Among the available numerical methods, finite element methods for state equations

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enjoy wide application (though other methods are also used of course). Many experts have made various contributions to the finite element methods for optimal control problems. Let us first mention two early papers devoted to linear-quadratic optimal control problems by Falk [11] and Geveci [12]. Moreover, Arada et al. [2] studied the numerical approximation of distributed nonlinear optimal control problems with pointwise constraints on the control. Meyer and Rösch [21] analyzed the discretization of the dimensional (2-d) elliptic optimal control problem. It is proved that these approximations have convergence order h^2. A posteriori error estimates for distributed convex optimal control problems and nonlinear optimal control problems have been obtained in [17, 18]. Huang et al. [15] constructed an adaptive multi-mesh finite element scheme for constrained distributed convex optimal control problem.

Compared with standard finite element methods, the mixed finite element methods have many advantages. In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. Some specialists have made many important works on some topic of mixed finite element method for linear optimal control problems.

Recently, in [8, 9], we obtained a posteriori error estimates and a priori error estimates of mixed finite element methods for quadratic optimal control problems. In [6, 7], we used the postprocessing projection operator to prove a quadratic superconvergence of the control by mixed finite element methods. We investigated the optimal control problem with the admissible control set, defined by

$$U_{ad} = \left\{ u \in L^2(\Omega) : a \le u \le b,\ \ a.e.\ \text{in}\, \Omega \right\},$$

where a and b are two real numbers, and obtained the superconvergence of $\mathcal{O}(h^{s+1})$ (for some $0<s\le 1$) for the control variable which is approximated by piecewise constant functions. Compared with it, our work changes the admissible set and we also get the same result.

For the constrained optimal control problem, the regularity of the optimal control is generally quite low. The goal of this paper is to investigate the superconvergence for the elliptic optimal control problem with a special admissible set which will be specified later.

We are concerned with the two dimensional elliptic optimal control problem

$$\min_{u\in U_{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \frac{1}{2}\|y-y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\}, \tag{1.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p} + a_0 y = u, \quad \boldsymbol{p} = -A(x)\mathbf{grad}y, \qquad x \in \Omega, \tag{1.2}$$

with the boundary condition

$$y = 0, \qquad x \in \partial\Omega, \tag{1.3}$$