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Travelling Wave Solutions to the Zhiber-Shabat and Related Equations Using Rational Hyperbolic Method

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Abstract. This paper presents the application of a new method for obtaining new exact solutions of some well-known nonlinear partial differential equations. The Rational Hyperbolic method is used for handling the Zhiber-Shabat equation and the related equations such as Liouville, Sinh-Gordon, Dodd-Bullough-Mikhailov and Tzitzeica-Dodd-Bullough equations. We show power of the Rational Hyperbolic method that is simple and effective for solving nonlinear partial differential equations.

AMS subject classifications: 35C07, 35G50

Key words: The rational hyperbolic method, Zhiber-Shabat, Liouville, Sinh-Gordon, Dodd-Bullough-Mikhailov and Tzitzeica-Dodd-Bullough equations.

1 Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena and have a significant role in several scientific and engineering fields. These equations appear in solid state physics [1], fluid mechanics [2], chemical kinetics [3], plasma physics [4], population models, nonlinear optics, propagation of fluxions in Josephson junctions, etc.

Analytical exact solutions to nonlinear partial differential equation play an important role in nonlinear science, since they can provide us much physical information and more insight into the physical aspects of the problem and thus lead to further applications. In recent years, quite a few methods for obtaining explicit travelling and

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solitary wave solutions of nonlinear evolutions equations have been proposed. A variety of powerful methods, such as inverse scattering method [5, 6], bilinear transformation [7], Bucklund and Darboux transformation [7–9], Hamiltonian structures [10], transformed rational [11], the tanh-sech method [12, 13], extended tanh method [14], Exp-Function method [15–18], the sine-cosine method [19–21], the Jacobi elliptic function method [22–24], F-expansion method [25, 26], Lie group analysis [27], He's variational iteration method [28, 29], He's homotopy perturbation method [29–31] and homogeneous balance method [32, 33] and so on.

Nonlinear equations play a major role in scientific fields. A class of equations, namely,

$$u_{xt} + f(u) = 0,$$

play a significant role in many scientific applications such as solid-state physics, nonlinear optics and quantum field theory where the function $f(u)$ takes many forms. In this paper, we investigate the nonlinear Zhiber-Shabat equation in the form

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0, \tag{1.1}$$

where p, q, and r are arbitrary constant [34–39]. When q=r=0 and p=1, we have:

$$u_{xt} + e^{u} = 0, \tag{1.2}$$

which is the well-known Liouville equation. When p=1, q=−1 and r=0, we have:

$$u_{xt} + e^{u} - e^{-u} = 0, \tag{1.3}$$

which is the well-known Sinh-Gordon equation. When q=0, p=1 and r=1, we have:

$$u_{xt} + e^{u} + e^{-2u} = 0, \tag{1.4}$$

which is the well-known Dodd-Bullough-Mikhailov equation. Moreover when p=0, q=−1 and r=1, we obtain the Tzitzeica-Dodd-Bullough equation:

$$u_{xt} - e^{-u} + e^{-2u} = 0. \tag{1.5}$$

The aforementioned equations play a significant role in many scientific applications such as solid-state physics, nonlinear optics, plasma physics, fluid dynamics, mathematical biology, nonlinear optics, dislocations in crystals, kink dynamics, and chemical kinetics, and quantum field theory [41–48]. Primarily, we introduce a wave variable η defined as $\eta=\lambda(x-\alpha t)$, where α is the wave speed. By using the traveling wave transformation $u(x,t)=U(\eta)$, we can write the Zhiber-Shabat equation (1.1) in the form

$$-\alpha\lambda^2 U'' + pe^{u} + qe^{-u} + re^{-2u} = 0. \tag{1.6}$$

From Eq. (1.2) we can write the Liouville equation in the form

$$-\alpha\lambda^2 U'' + e^{u} = 0. \tag{1.7}$$

and from Eq. (1.3) we can write the Sinh-Gordon equation in the form

$$-\alpha\lambda^2 U'' + e^u - e^{-u} = 0. \tag{1.8}$$

Similarly, the Dodd-Bullough-Mikhailov equation (1.4) can be written in the form

$$-\alpha\lambda^2 U'' + e^u + e^{-2u} = 0, \tag{1.9}$$

and the Tzitzeica-Dodd-Bullough equation (1.5) becomes

$$-\alpha\lambda^2 U'' - e^{-u} + e^{-2u} = 0, \tag{1.10}$$

where prime denotes the differential with respect to η.

2 Summary of hyperbolic-Rational method

It is appropriate to introduce rational hyperbolic functions methods by setting

$$V(\eta) = \frac{a_1}{1 + a_2 f(\eta)}, \tag{2.1}$$

$$V(\eta) = \frac{a_1}{1 + a_2 f^2(\eta)}, \tag{2.2}$$

where a_1, a_2 and λ are parameters that will be determined, and

$$f(x,t) = \begin{cases} \text{sech}(\eta), \\ \tanh(\eta), \\ \coth(\eta). \end{cases} \tag{2.3}$$

The rational hyperbolic functions methods can be applied directly in a straightforward manner. We then collect the coefficients of the resulting hyperbolic functions and setting it equal to zero, and solve the resulting equations to determine the parameters a_1 and a_2.

The second rational hyperbolic functions methods can be expressed in the forms

$$V(\eta) = \frac{a_1 + a_2 f(\eta)}{1 + a_3 f(\eta)}, \tag{2.4}$$

$$V(\eta) = \frac{a_1 + a_2 f^2(\eta)}{1 + a_3 f^2(\eta)}, \tag{2.5}$$

where a_1, a_2 and a_3 are parameters to be determined, and $f(\eta)$ is defined in Eq. (2.3). The second rational hyperbolic functions methods can be applied directly as assumed before.

3 New application of methods

We now consider the Zhiber-Shabat, Liouville, Sinh-Gordon, Dodd-Bullough-Mikhailov and Tzitzeica-Dodd-Bullough equations using the Rational-Tanh Method.

3.1 Using Tanh-Rational methods for the Zhiber-Shabat equation

By using the transformation

$$V(\eta) = e^{U(\eta)}, \qquad U(\eta) = \ln\left(V(\eta)\right), \tag{3.1}$$

Eq. (1.6) becomes an partial differential equation of the form

$$-\alpha\lambda^2\left(V''V - V'^2\right) + pV^3 + qV + r = 0. \tag{3.2}$$

In this case, we use the second rational hyperbolic method with $f(\eta) = \tanh(\eta)$. Substituting Eq. (2.5) into Eq. (3.2) with $f(\eta) = \tanh(\eta)$ and using Maple, we get a system of algebraic equation, for a_1, a_2, a_3 and λ:

$$\begin{aligned}
\tanh^0(\eta) &\to 2\alpha\lambda^2 a_3 a_1^2 + r - 2\alpha\lambda^2 a_2 a_1 + r a_1 + r a_1^3 = 0,\\
\tanh^2(\eta) &\to -8\alpha\lambda^2 a_3 a_1^2 + r a_2 + 8\alpha\lambda^2 a_2 a_1 + r a_1^3 a_3 + 3 r a_1 a_3\\
&\quad + 4 r a_3 + 2\alpha\lambda^2 a_2^2 - 2\alpha\lambda^2 a_3^2 a_1^2 + 3 r a_1^2 a_3 = 0,\\
\tanh^4(\eta) &\to 6\alpha\lambda^2 a_3 a_1^2 + 3 r a_2^2 a_1 + 6 r a_3^2 + 3 r a_1 a_3^3 - 6\alpha\lambda^2 a_2 a_1\\
&\quad + 6\alpha\lambda^2 a_2^2 a_3 + 3 r a_2 a_3 + 3 r a_1^2 a_2 a_3 - 6\alpha\lambda^2 a_3^2 a_2 a_1 = 0,\\
\tanh^6(\eta) &\to r a_2^3 + r a_1 a_3^3 + 2\alpha\lambda^2 a_3^2 a_1^2 - 8\alpha\lambda^2 a_2^2 a_3 + 8\alpha\lambda^2 a_3^2 a_1 a_2\\
&\quad + 3 r a_3^2 a_2 + 3 r a_2^2 a_1 a_3 + 4 r a_1^3 - 2\alpha\lambda^2 a_2^2 = 0,\\
\tanh^8(\eta) &\to r a_2^3 a_3 + 2\alpha\lambda^2 a_2^2 a_3 - 2\alpha\lambda^2 a_3^2 a_1 a_2 + r a_3^3 a_2 + r a_3^4 = 0.
\end{aligned}$$

Solving the resulting set of equation with the aid of Maple and introducing the parameter ξ, we obtain a different set of results for each ξ:

$$\alpha = -\frac{q^2\xi^2 + 4rq\xi + 3r^2}{4r\lambda^2\xi^2}, \quad a_1 = \frac{\xi}{2}, \quad a_2 = -\frac{q^2\xi^2 + 4rq\xi + 3r^2}{2rp\xi^2}, \quad a_3 = 0. \tag{3.3}$$

Inserting these values into Eq. (2.5), we obtain:

$$V(\eta) = \frac{\xi}{2} - \frac{q^2\xi^2 + 4rq\xi + 3r^2}{2rp\xi^2}\tanh^2(\eta). \tag{3.4}$$

From Eq. (2.4), we can obtain $U(\eta)$:

$$U(\eta) = \ln\left(\frac{\xi}{2} - \frac{q^2\xi^2 + 4rq\xi + 3r^2}{2rp\xi^2}\tanh^2(\eta)\right). \tag{3.5}$$

Substituting $\eta = \lambda(x - \lambda t)$ into this result, we obtain:

$$u(x,t) = \ln\left(\frac{\xi}{2} - \frac{q^2\xi^2 + 4rq\xi + 3r^2}{2rp\xi^2}\tanh^2\left(\lambda\left(x - \alpha t\right)\right)\right). \tag{3.6}$$

Moreover, from Eq. (3.1), we know

$$\alpha = -\frac{q^2\xi^2 + 4rq\xi + 3r^2}{4r\lambda^2\xi^2}.$$

Consequently, we have

$$u\left(x,t\right)=\ln\left(\frac{\xi}{2}-\frac{q^2\xi^2+4rq\xi+3r^2}{2rp\xi^2}\tanh^2\left(\lambda x+\frac{q^2\xi^2+4rq\xi+3r^2}{4r\lambda\xi^2}t\right)\right). \tag{3.7}$$

Considering to parameter γ, we find ξ that contains three cases as follow:

$$\xi_1=\frac{\gamma}{6rp}+\frac{2q\left(6pr^2+q^3\right)}{3rp\gamma}+\frac{q^2}{3rp}, \tag{3.8a}$$

$$\xi_2=\frac{\gamma}{12rp}-\frac{q\left(6pr^2+q^3\right)}{3rp\gamma}+\frac{q^2}{3rp}+\frac{I\sqrt{3}}{2}\Big(\frac{\gamma}{6rp}-\frac{2q\left(6pr^2+q^3\right)}{3rp\gamma}\Big), \tag{3.8b}$$

$$\xi_3=-\frac{\gamma}{12rp}-\frac{q\left(6pr^2+q^3\right)}{3rp\gamma}+\frac{q^2}{3rp}-\frac{I\sqrt{3}}{2}\Big(\frac{\gamma}{6rp}-\frac{2q\left(6pr^2+q^3\right)}{3rp\gamma}\Big), \tag{3.8c}$$

where

$$\gamma=\left(72q^3r^2p+108r^4p^2+8q^6+12r^3p^2\sqrt{\frac{12q^3}{p}+81r^2}\right)^{\frac{1}{3}}. \tag{3.8d}$$

If we substitute ξ_1, ξ_2 and ξ_3 into Eq. (3.7), we obtain three different solutions for the Zhiber-Shabat equation. Wazwaz finds some solution for the Zhiber-Shabat equation [36] when $p=q=r=1$ but we find three different exact solutions for the Zhiber-Shabat equation in general condition.

If we substitute $p=q=r=1$ into Eq. (3.7), then we obtain:

$$u(x,t)=1.073949518-1.756277322\tanh^2\Big(\lambda x+0.8781386608\frac{t}{\lambda}\Big), \tag{3.9a}$$

$$\begin{aligned}u(x,t)=&-.2869747589+.1844947039I+\Big(0.6281386630\\&-1.346036104I\Big)\tanh^2\Big(\lambda x+(0.3140693315-0.6730180522I)\frac{t}{\lambda}\Big),\end{aligned} \tag{3.9b}$$

$$\begin{aligned}u(x,t)=&-.2869747589-.1844947039I+\Big(.6281386630\\&+1.346036104I\Big)\tanh^2\Big(\lambda x+(0.3140693315+0.6730180522I)\frac{t}{\lambda}\Big).\end{aligned} \tag{3.9c}$$

3.2 Using the rational hyperbolic method for the Dodd-Bullough-Mikhailov equation

In this case, we consider the Dodd-Bullough-Mikhailov equation using the rational tanh method, as explained above. By using the transformation

$$V\left(\eta\right)=e^U,\qquad U\left(\eta\right)=\ln\left(V\left(\eta\right)\right), \tag{3.10}$$

Eq. (1.9) becomes an partial differential equation of the form

$$-\alpha\lambda^2\left(V''V-V'^2\right)+V^3+1=0. \tag{3.11}$$

Substituting Eq. (2.5) into Eq. (3.11) with $f(\eta) = \tanh(\eta)$ and using Maple, we get a system of algebraic equation, for a_1, a_2, a_3, α, λ:

$$\begin{aligned}
&\tanh^0(\eta) \to 2\alpha\lambda^2 a_3 a_1^2 + 1 - 2\alpha\lambda^2 a_2 a_1 + a_1^3 = 0,\\
&\tanh^2(\eta) \to -8\alpha\lambda^2 a_3 a_1^2 + 8\alpha\lambda^2 a_2 a_1 + a_1^3 a_3 + 4a_3 + 2\alpha\lambda^2 a_2^2 - 2\alpha\lambda^2 a_3^2 a_1^2 + 3a_1^2 a_3 = 0,\\
&\tanh^4(\eta) \to 6\alpha\lambda^2 a_3 a_1^2 + 3a_2^2 a_1 + 6a_3^2 - 6\alpha\lambda^2 a_2 a_1 + 6\alpha\lambda^2 a_2^2 a_3 + 3a_1^2 a_2 a_3 - 6\alpha\lambda^2 a_3^2 a_2 a_1 = 0,\\
&\tanh^6(\eta) \to a_2^3 + 2\alpha\lambda^2 a_3^2 a_1^2 - 8\alpha\lambda^2 a_2^2 a_3 + 8\alpha\lambda^2 a_3^2 a_1 a_2 + 3a_2^2 a_1 a_3 + 4a_3^3 - 2\alpha\lambda^2 a_2^2 = 0,\\
&\tanh^8(\eta) \to a_2^3 a_3 + 2\alpha\lambda^2 a_2^2 a_3 - 2\alpha\lambda^2 a_3^2 a_1 a_2 + a_3^3 a_2 + a_3^4 = 0.
\end{aligned}$$

Solving the resulting set of equations with the aid of Maple, we can distinguish different cases.

Case 1:

$$\alpha = -\frac{3}{4\lambda^2}, \quad a_1 = \frac{1}{2}, \quad a_2 = -\frac{3}{2}, \quad a_3 = 0. \tag{3.12}$$

Inserting these values into Eq. (2.5), we obtain

$$V(\eta) = \frac{1}{2} - \frac{3}{2}\tanh^2(\eta). \tag{3.13}$$

From Eq. (3.10), we can obtain $U(\eta)$:

$$U(\eta) = \ln\Big(\frac{1}{2} - \frac{3}{2}\tanh^2(\eta)\Big). \tag{3.14}$$

Substituting $\eta=\lambda(x-\alpha t)$ into this result, we obtain

$$u(x,t) = \ln\Big(\frac{1}{2} - \frac{3}{2}\tanh^2(\lambda(x-\alpha t))\Big). \tag{3.15}$$

Moreover, from Eq. (3.12), we know $\alpha=-3/(4\lambda^2)$ and then we have

$$u(x,t) = \ln\Big(\frac{1}{2} - \frac{3}{2}\tanh^2\Big(\lambda x + \frac{3}{4\lambda}t\Big)\Big). \tag{3.16}$$

Case 2:

$$\alpha = \frac{3+3I\sqrt{3}}{8\lambda^2}, \quad a_1 = -\frac{1}{4} - \frac{1}{4}I\sqrt{3}, \quad a_2 = \frac{3}{4} + \frac{3}{4}I\sqrt{3}, \quad a_3 = 0, \tag{3.17a}$$

$$\alpha = \frac{3-3I\sqrt{3}}{8\lambda^2}, \quad a_1 = -\frac{1}{4} + \frac{1}{4}I\sqrt{3}, \quad a_2 = \frac{3}{4} - \frac{3}{4}I\sqrt{3}, \quad a_3 = 0. \tag{3.17b}$$

Inserting these values into Eq. (2.5) gives

$$V(\eta) = -\frac{1}{4} - \frac{1}{4}I\sqrt{3} + \frac{3}{4}\Big(1 + I\sqrt{3}\Big)\tanh^2(\eta), \tag{3.18a}$$

$$V(\eta) = -\frac{1}{4} + \frac{1}{4}I\sqrt{3} + \frac{3}{4}\Big(1 - I\sqrt{3}\Big)\tanh^2(\eta). \tag{3.18b}$$

From Eq. (3.10), we can obtain $U(\eta)$:

$$U(\eta) = \ln\Big(-\frac{1}{4} - \frac{1}{4}I\sqrt{3} + \frac{3}{4}(1 + I\sqrt{3})\tanh^2(\eta)\Big), \tag{3.19a}$$

$$U(\eta) = \ln\Big(-\frac{1}{4} + \frac{1}{4}I\sqrt{3} + \frac{3}{4}(1 - I\sqrt{3})\tanh^2(\eta)\Big). \tag{3.19b}$$

Substituting $\eta = \lambda(x - \alpha t)$ into this result gives

$$u(x,t) = \ln\Big(-\frac{1}{4} - \frac{1}{4}I\sqrt{3} + \frac{3}{4}(1 + I\sqrt{3})\tanh^2(\lambda(x - \alpha t))\Big), \tag{3.20a}$$

$$u(x,t) = \ln\Big(-\frac{1}{4} + \frac{1}{4}I\sqrt{3} + \frac{3}{4}(1 - I\sqrt{3})\tanh^2(\lambda(x - \alpha t))\Big). \tag{3.20b}$$

Moreover, substituting α from Eq. (3.18a) yields

$$u(x,t) = \ln\Big(\frac{1}{2} + \frac{1}{2}I\sqrt{3} - \frac{3}{2}\Big(\frac{1}{2} + \frac{1}{2}I\sqrt{3}\Big)\operatorname{sech}^2\Big(\lambda x - \frac{3 + 3I\sqrt{3}}{8\lambda}t\Big)\Big), \tag{3.21}$$

$$u(x,t) = \ln\Big(\frac{1}{2} - \frac{1}{2}I\sqrt{3} - \frac{3}{2}\Big(\frac{1}{2} - \frac{1}{2}I\sqrt{3}\Big)\operatorname{sech}^2\Big(\lambda x - \frac{3 - 3I\sqrt{3}}{8\lambda}t\Big)\Big). \tag{3.22}$$

3.3 Using rational hyperbolic methods for the Tzitzeica-Dodd-Bullough equation

In this case, we consider the Tzitzeica-Dodd-Bullough equation using rational hyperbolic method, which was explained above: By using the transformation

$$V(\eta) = e^U, \qquad U(\eta) = \ln(V(\eta)). \tag{3.23}$$

Eq. (1.10) becomes an partial differential equation, which reads :

$$-\alpha\lambda^2\left(V''V - V'^2\right) - V + 1 = 0. \tag{3.24}$$

Case 1:

Substituting Eq. (2.4) into Eq. (3.24) with $f(\eta) = \tanh(\eta)$, and using Maple, we get a system of algebraic equation, for a_1, a_2, a_3, α, λ:

$$\begin{aligned}
&\tanh^0(\eta) \to 1 - a_1 - \alpha\lambda^2 a_3^2 a_1^2 + \alpha\lambda^2 a_2^2 = 0,\\
&\tanh^1(\eta) \to -a_2 + 2\alpha\lambda^2 a_2 a_1 + 2\alpha\lambda^2 a_2^2 a_3 - 3a_1 a_2 + 4a_3 - 2\alpha\lambda^2 a_3^2 a_2 a_1 - 2\alpha\lambda^2 a_3 a_1^2 = 0,\\
&\tanh^2(\eta) \to -3a_1 a_3^2 - 3a_2 a_3 + 6a_3^2 = 0,\\
&\tanh^3(\eta) \to 2\alpha\lambda^2 a_3^2 a_2 a_1 + 2\alpha\lambda^2 a_3 a_1^2 - 2\alpha\lambda^2 a_3 a_2^2 - 3a_2 a_3^2 - 2\alpha\lambda^2 a_2 a_1 + 4a_3^3 - a_1 a_3^3 = 0,\\
&\tanh^4(\eta) \to \alpha\lambda^2 a_3^2 a_1^2 + a_3^2 - \alpha\lambda^2 a_2^2 - a_3^3 a_2 = 0.
\end{aligned}$$

Solving the set of equation with the aid of Maple, we obtain:

$$\alpha = -\frac{1}{4\lambda^2}, \qquad a_2 = -a_1 + 2, \qquad a_3 = 1, \tag{3.25a}$$

$$\alpha = -\frac{1}{4\lambda^2}, \qquad a_2 = a_1 - 2, \qquad a_3 = -1. \tag{3.25b}$$

Inserting these values into Eq. (2.4), we obtain:

$$V(\eta) = -\frac{(a_1-2)\tanh(\eta) - a_1}{1+\tanh(\eta)}, \tag{3.26a}$$

$$V(\eta) = -\frac{(a_1-2)\tanh(\eta) + a_1}{-1+\tanh(\eta)}. \tag{3.26b}$$

From Eq. (3.23), we can obtain :

$$U(\eta) = \ln\Big(-\frac{(a_1-2)\tanh(\eta) - a_1}{1+\tanh(\eta)}\Big), \tag{3.27a}$$

$$U(\eta) = \ln\Big(-\frac{(a_1-2)\tanh(\eta) + a_1}{-1+\tanh(\eta)}\Big). \tag{3.27b}$$

Substituting $\eta = \lambda x + t/(4\lambda)$ into this result, we obtain:

$$u(x,t) = \ln\Big(-\frac{(a_1-2)\tanh\big(\lambda x + \frac{1}{4\lambda}t\eta\big) - a_1}{1+\tanh\big(\lambda x + \frac{1}{4\lambda}t\big)}\Big), \tag{3.28a}$$

$$u(x,t) = \ln\Big(-\frac{(a_1-2)\tanh\big(\lambda x + \frac{1}{4\lambda}t\big) + a_1}{-1+\tanh\big(\lambda x + \frac{1}{4\lambda}t\big)}\Big). \tag{3.28b}$$

If we substitute $a_1 = 2$ then we obtain:

$$u(x,t) = \ln\Big(\frac{2}{1+\tanh\big(\lambda x + \frac{1}{4\lambda}t\big)}\Big), \tag{3.29a}$$

$$u(x,t) = \ln\Big(\frac{2}{1-\tanh\big(\lambda x + \frac{1}{4\lambda}t\big)}\Big). \tag{3.29b}$$

Case 2:

Substituting Eq. (2.5) into Eq. (3.24) with $f(\eta) = \tanh(\eta)$, and using Maple, we get a system of algebraic equation, for a_1, a_2, a_3, α, λ:

$$\begin{aligned}
&\tanh^0(\eta) \to 1 - a_1 + \alpha\lambda^2 a_3 a_1^2 - 2\alpha\lambda^2 a_2 a_1 = 0,\\
&\tanh^2(\eta) \to -a_2 + 4a_3 + 8\alpha\lambda^2 a_2 a_1 - 8\alpha\lambda^2 a_3 a_1^2 - 3a_1 a_3 - 2\alpha\lambda^2 a_3^2 a_1^2 + 2\alpha\lambda^2 a_2^2 = 0,\\
&\tanh^4(\eta) \to 6\alpha\lambda^2 a_2 a_3^2 - 3a_2 a_3 - 6\alpha\lambda^2 a_3^2 a_2 a_1 - 3a_1 a_3^2 + 6a_3^2 + 6\alpha\lambda^2 a_1^2 a_3 - 6\alpha\lambda^2 a_2 a_1 = 0,\\
&\tanh^6(\eta) \to -8\alpha\lambda^2 a_3^2 a_2 a_1 - 3a_2 a_3^2 + 4a_3^3 - a_1 a_3^3 + 2\alpha\lambda^2 a_3^2 a_1^2 + 8\alpha\lambda^2 a_3 a_1 a_2^2 = 0,\\
&\tanh^8(\eta) \to 2\alpha\lambda^2 a_2^2 a_3 + a_3^4 - a_3^3 a_2 - 2\alpha\lambda^2 a_3^2 a_1 a_2 = 0.
\end{aligned}$$

Solving the resulting set of equations with the aid of Maple, we obtain

$$\alpha = \frac{1-2a_1}{4\lambda^2 a_1^2}, \quad a_2 = \frac{a_1}{1-2a_1}, \quad a_3 = -1. \tag{3.30}$$

Inserting these values into Eq. (2.5), we obtain

$$V(\eta) = \frac{a_1 - \frac{a_1 \tanh^2(\eta)}{-1+2a_1}}{1-\tanh^2(\eta)}. \tag{3.31}$$

From Eq. (3.23), we can obtain $U(\eta)$:

$$U(\eta) = \ln \frac{a_1 - \frac{a_1 \tanh^2(\eta)}{-1+2a_1}}{1-\tanh^2(\eta)}. \tag{3.32}$$

Substituting

$$\eta = \lambda x + \frac{1-2a_1}{4\lambda a_1^2} t$$

into (3.32) gives

$$u(x,t) = \ln \frac{a_1 - \frac{a_1 \tanh^2\left(\lambda x + \frac{1-2a_1}{4\lambda a_1^2} t\right)}{-1+2a_1}}{1-\tanh^2\left(\lambda x + \frac{1-2a_1}{4\lambda a_1^2} t\right)}. \tag{3.33}$$

If we substitute $a_1 = 2$, then we obtain:

$$u(x,t) = \ln \frac{2 - \frac{2}{3}\tanh^2\left(-\lambda x + \frac{3}{16\lambda} t\right)}{1-\tanh^2\left(-\lambda x + \frac{3}{16\lambda} t\right)}. \tag{3.34}$$

3.4 Using the rational hyperbolic method for the Liouville equation

In this case, we consider the Liouville equation using the rational tanh method, as explained above. By using the transformation

$$V(\eta) = e^U, \qquad U(\eta) = \ln(V(\eta)), \tag{3.35}$$

Eq. (1.7) becomes an partial differential equation of the form

$$-\alpha\lambda^2\left(V''V - V'^2\right) + V^3 = 0. \tag{3.36}$$

Substituting Eq. (2.5) into Eq. (3.36) with $f(\eta) = \tanh(\eta)$, and using Maple, we get a system of algebraic equation, for a_1, a_2, a_3, α, λ:

$$\begin{aligned}
&\tanh^0(\eta) \to 2\alpha\lambda^2 a_3 a_1^2 - 2\alpha\lambda^2 a_2 a_1 + a_1^3 = 0,\\
&\tanh^2(\eta) \to -8\alpha\lambda^2 a_3 a_1^2 + 8\alpha\lambda^2 a_2 a_1 + a_1^3 a_3 + 2\alpha\lambda^2 a_2^2 - 2\alpha\lambda^2 a_3^2 a_1^2 + 3a_1^2 a_3 = 0,\\
&\tanh^4(\eta) \to 6\alpha\lambda^2 a_3 a_1^2 + 3a_2^2 a_1 - 6\alpha\lambda^2 a_2 a_1 + 6\alpha\lambda^2 a_2^2 a_3 + 3a_1^2 a_2 a_3 - 6\alpha\lambda^2 a_3^2 a_2 a_1 = 0,\\
&\tanh^6(\eta) \to a_2^3 + 2\alpha\lambda^2 a_3^2 a_1^2 - 8\alpha\lambda^2 a_2^2 a_3 + 8\alpha\lambda^2 a_3^2 a_1 a_2 + 3a_2^2 a_1 a_3 - 2\alpha\lambda^2 a_2^2 = 0,\\
&\tanh^8(\eta) \to a_2^3 a_3 + 2\alpha\lambda^2 a_2^2 a_3 - 2\alpha\lambda^2 a_3^2 a_1 a_2 0.
\end{aligned}$$

Solving the set of equation with the aid of Maple, we obtain

$$a_1 = -2\alpha\lambda^2, \quad a_2 = 2\alpha\lambda^2, \quad a_3 = 0. \tag{3.37}$$

Inserting these values into Eq. (2.5) gives

$$V(\eta) = -2\alpha\lambda^2 + 2\alpha\lambda^2 \tanh^2(\eta). \tag{3.38}$$

From Eq. (3.35), we can obtain $U(\eta)$:

$$U(\eta) = \ln\left(-2\alpha\lambda^2 + 2\alpha\lambda^2 \tanh^2(\eta)\right). \tag{3.39}$$

Substituting $\eta = \lambda(x - \alpha t)$ into this result yields

$$u(x,t) = \ln\left(-2\alpha\lambda^2 + 2\alpha\lambda^2 \tanh^2(\lambda(x - \alpha t))\right). \tag{3.40}$$

3.5 Using the rational tanh methods for the Sinh-Gordon equation

In this case, we consider the Sinh-Gordon equation using the rational tanh method, as explained above. By using the transformation

$$V(\eta) = e^U, \qquad U(\eta) = \ln(V(\eta)), \tag{3.41}$$

Eq. (1.8) becomes an partial differential equation of the form

$$-\alpha\lambda^2\left(V''V - V'^2\right) + V^3 + V = 0. \tag{3.42}$$

Substituting Eq. (2.5) into Eq. (3.42) with $f(\eta) = \tanh(\eta)$, and using Maple, we get a system of algebraic equation, for a_1, a_2, a_3, α, λ:

$$\begin{aligned}
&\tanh^0(\eta) \to 2\alpha\lambda^2 a_3 a_1^2 - 2\alpha\lambda^2 a_2 a_1 + a_1^3 + a_1 = 0,\\
&\tanh^2(\eta) \to -8\alpha\lambda^2 a_3 a_1^2 + 8\alpha\lambda^2 a_2 a_1 + a_1^3 a_3 + a_2 + 2\alpha\lambda^2 a_2^2 - 2\alpha\lambda^2 a_3^2 a_1^2 + 3a_1^2 a_3 + 3a_1^2 a_2 = 0,\\
&\tanh^4(\eta) \to 6\alpha\lambda^2 a_3 a_1^2 + 3a_2^2 a_1 + 3a_3^2 a_1 - 6\alpha\lambda^2 a_2 a_1 + 6\alpha\lambda^2 a_2^2 a_3 + 3a_2 a_3 + 3a_1^2 a_2 a_3\\
&\qquad\qquad - 6\alpha\lambda^2 a_3^2 a_2 a_1 = 0,\\
&\tanh^6(\eta) \to a_2^3 + 2\alpha\lambda^2 a_3^2 a_1^2 - 8\alpha\lambda^2 a_2^2 a_3 + 8\alpha\lambda^2 a_3^2 a_1 a_2 + 3a_2^2 a_1 a_3 + a_1 a_3^3 + 3a_3^2 a_2\\
&\qquad\qquad - 2\alpha\lambda^2 a_2^2 = 0,\\
&\tanh^8(\eta) \to a_2^3 a_3 + 2\alpha\lambda^2 a_2^2 a_3 - 2\alpha\lambda^2 a_3^2 a_1 a_2 + a_3^3 a_2 = 0.
\end{aligned}$$

Solving the set of equation with the aid of Maple, we obtain:

$$\alpha = \frac{I}{2\lambda^2}, \qquad a_1 = 0, \qquad a_2 = I, \qquad a_2 = 0, \tag{3.43}$$

$$\alpha = -\frac{I}{2\lambda^2}, \qquad a_1 = 0, \qquad a_2 = -I, \qquad a_2 = 0. \tag{3.44}$$

Inserting these values into Eq. (2.5), we obtain:

$$V(\eta) = I\tanh^2(\eta), \qquad V(\eta) = -I\tanh^2(\eta). \tag{3.45a}$$

From Eq. (3.41), we can obtain $U(\eta)$:

$$U(\eta) = \ln\Big(I\tanh^2(\eta)\Big), \qquad U(\eta) = \ln\Big(-I\tanh^2(\eta)\Big). \tag{3.46a}$$

Substituting $\eta = \lambda(x - \alpha t)$ into this result, we obtain:

$$u(x,t) = \ln\Big(I\tanh^2\left(\lambda\left(x - \alpha t\right)\right)\Big), \tag{3.47a}$$

$$u(x,t) = \ln\Big(-I\tanh^2\left(\lambda\left(x - \alpha t\right)\right)\Big). \tag{3.47b}$$

Moreover, substituting α from Eq. (3.43):

$$u(x,t) = \ln\Big(I\tanh^2\Big(\lambda x - \frac{I}{2\lambda}t\Big)\Big), \tag{3.48a}$$

$$u(x,t) = \ln\Big(-I\tanh^2\Big(\lambda x + \frac{I}{2\lambda}t\Big)\Big). \tag{3.48b}$$

4 Discussion and conclusions

The Zhiber-Shabat equation and the related equations including the Liouville equation, the sinh-Gordon equation, the Dodd-Bullough-Mikhailov equation, and the Tzitzeica-Dodd-Bullough equation were investigated by using the rational hyperbolic method. Travelling wave solutions were established by using this method and several of the obtained solutions are entirely new. The main goals of this work, i.e., to determine travelling wave solutions and to emphasize the power of the rational hyperbolic method, have been achieved. We handle the nonlinear partial differential equations by solving ordinary differential equations where the balancing process is used to determine the solution as a polynomial in tanh(ln).

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