Superconvergence of Rectangular Mixed Finite Element Methods for Constrained Optimal Control Problem

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Abstract. We investigate the superconvergence properties of the constrained quadratic elliptic optimal control problem which is solved by using rectangular mixed finite element methods. We use the lowest order Raviart-Thomas mixed finite element spaces to approximate the state and co-state variables and use piecewise constant functions to approximate the control variable. We obtain the superconvergence of $\mathcal{O}(h^{1+s})$ $(0<s\leq1)$ for the control variable. Finally, we present two numerical examples to confirm our superconvergence results.

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1   Introduction

In this paper, we focus on the superconvergence properties of rectangular mixed finite element methods for linear elliptic optimal control problem. Optimal control problems are playing increasingly important role in the design of modern life. They have various applications in the operation of physical, social, and economic processes. Among the available numerical methods, finite element methods for state equations

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enjoy wide application (though other methods are also used of course). Many experts have made various contributions to the finite element methods for optimal control problems. Let us first mention two early papers devoted to linear-quadratic optimal control problems by Falk [11] and Geveci [12]. Moreover, Arada et al. [2] studied the numerical approximation of distributed nonlinear optimal control problems with pointwise constraints on the control. Meyer and Rösch [21] analyzed the discretization of the dimensional (2-d) elliptic optimal control problem. It is proved that these approximations have convergence order h^2. A posteriori error estimates for distributed convex optimal control problems and nonlinear optimal control problems have been obtained in [17,18]. Huang et al. [15] constructed an adaptive multi-mesh finite element scheme for constrained distributed convex optimal control problem.

Compared with standard finite element methods, the mixed finite element methods have many advantages. In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. Some specialists have made many important works on some topic of mixed finite element method for linear optimal control problems.

Recently, in [8,9], we obtained a posteriori error estimates and a priori error estimates of mixed finite element methods for quadratic optimal control problems. In [6,7], we used the postprocessing projection operator to prove a quadratic superconvergence of the control by mixed finite element methods. We investigated the optimal control problem with the admissible control set, defined by

$$U_{ad} = \left\{ u \in L^2(\Omega) : a \le u \le b, \ a.e. \ \text{in} \ \Omega \right\},$$

where a and b are two real numbers, and obtained the superconvergence of $\mathcal{O}(h^{s+1})$ (for some $0 < s \le 1$) for the control variable which is approximated by piecewise constant functions. Compared with it, our work changes the admissible set and we also get the same result.

For the constrained optimal control problem, the regularity of the optimal control is generally quite low. The goal of this paper is to investigate the superconvergence for the elliptic optimal control problem with a special admissible set which will be specified later.

We are concerned with the two dimensional elliptic optimal control problem

$$\min_{u \in U_{ad}} \left\{ \frac{1}{2} \| \boldsymbol{p} - \boldsymbol{p}_d \|^2 + \frac{1}{2} \| y - y_d \|^2 + \frac{\nu}{2} \| u \|^2 \right\}, \tag{1.1}$$

subject to the state equation

$$\text{div} \boldsymbol{p} + a_0 y = u, \quad \boldsymbol{p} = -A(x) \mathbf{grad} y, \qquad x \in \Omega, \tag{1.2}$$

with the boundary condition

$$y = 0, \qquad x \in \partial \Omega, \tag{1.3}$$

where Ω is a rectangular domain, p_d and y_d are two known functions, p and y are the state variables, u is the control variable, and $\nu > 0$ is a constant. We denote $L^2(\Omega)$ norm by $\|\cdot\|$ and the set of admissible controls by U_{ad}, where

$$U_{ad} = \left\{ u \in L^2(\Omega) : \int_{\Omega} u(x)dx \geq 0 \right\}. \tag{1.4}$$

This paper is organized as follows. In next section, we construct a discretized scheme for the optimal control problem (1.1)-(1.3). In section 3, we consider the local L^2 superconvergence of the mixed finite element approximations for the control problem. In section 4, we carry out the global L^2 superconvergence of rectangular lowest order Raviart-Thomas mixed finite element methods. In section 5, two numerical examples are presented to demonstrate our theoretical results. Finally, we give the conclusions and comment on possible future work in section 6.

2   Mixed methods for optimal control problem

We shall construct a discretized scheme for the optimal control problem (1.1)-(1.3) by using mixed finite element methods and give its equivalent optimality conditions.

At first, we make the following assumption for the coefficient matrix $A(x)$.

(A1) The coefficient matrix function $A(x)=(a_{ij}(x))$ is symmetric with $a_{ij}(x)\in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \qquad \forall\, (\xi,x) \in R^2 \times \overline{\Omega}, \qquad c_* > 0.$$

Next, we introduce the co-state elliptic equation

$$\text{div}\boldsymbol{q} + a_0 z = y - y_d, \quad \boldsymbol{q} = -A(x)(\mathbf{grad}z + \boldsymbol{p} - \boldsymbol{p}_d), \qquad x \in \Omega, \tag{2.1}$$

with the boundary condition

$$z = 0, \qquad x \in \partial\Omega. \tag{2.2}$$

Lemma 2.1. *Assume that Ω is a convex polygonal domain. Let the state variable y and the co-state variable z be the variational solutions of (1.2)-(1.3) and (2.1)-(2.2), respectively. Then, there exists $0 < s_0 \leq 1$ such that for $0 < s < s_0$,*

$$\|y\|_{H^{2+s}(\Omega)} \leq C\|u\|_{H^s(\Omega)}, \tag{2.3}$$

$$\|z\|_{H^{2+s}(\Omega)} \leq C\|y\|_{H^s(\Omega)}. \tag{2.4}$$

Remark 2.1. It is known (see e.g., [3]) that $s_0 = \min\{1, \pi/\omega - 1\}$ when both the state equation and co-state equation are Laplace equation, where ω is the radian measure of the largest corner of the domain Ω ($\omega < \pi$). If the state equation and the co-state equation are variable coefficient equations, then s_0 is dependent on ω and the eigenvalue of the coefficient matrix $A(x)$ at the corner points.

Thus, we make the following realistic assumption (**A2**)

$$u \in W^{1,\infty}(\Omega), \quad y, z \in H^{2+s}(\Omega), \quad \text{for} \quad 0 < s \leq 1. \tag{2.5}$$

We shall obtain superconvergence results by using the operator interpolation technique. Let

$$\boldsymbol{V} = H(\text{div}; \Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega)^2, \ \text{div} \boldsymbol{v} \in L^2(\Omega) \right\}, \qquad W = L^2(\Omega).$$

The Hilbert space $\boldsymbol{V}$ is equipped with the following norm:

$$\|\boldsymbol{v}\|_{\text{div}} = \|\boldsymbol{v}\|_{H(\text{div};\Omega)} = \left(\|\boldsymbol{v}\|^2 + \|\text{div} \boldsymbol{v}\|^2 \right)^{\frac{1}{2}}.$$

Then, the weak formulation of the optimal control problem (1.1)-(1.3) is to find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \left\{ \frac{1}{2} \| \boldsymbol{p} - \boldsymbol{p}_d \|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\nu}{2} \|u\|^2 \right\}, \tag{2.6}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \text{div} \boldsymbol{v}) = 0, \qquad \forall \, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.7}$$

$$(\text{div} \boldsymbol{p}, w) + (a_0 y, w) = (u, w), \qquad \forall \, w \in W, \tag{2.8}$$

where the inner product in $L^2(\Omega)$ or $\left(L^2(\Omega)\right)^2$ is denoted by $(\cdot, \cdot)$. It is well known (see, e.g., [9]) that the convex control problem (2.6)-(2.8) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet (2.6)-(2.8) if and only if there exists a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \text{div} \boldsymbol{v}) = 0, \qquad \forall \, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.9}$$

$$(\text{div} \boldsymbol{p}, w) + (a_0 y, w) = (u, w), \qquad \forall \, w \in W, \tag{2.10}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \text{div} \boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \qquad \forall \, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.11}$$

$$(\text{div} \boldsymbol{q}, w) + (a_0 z, w) = (y - y_d, w), \qquad \forall \, w \in W, \tag{2.12}$$

$$(z + \nu u, \tilde{u} - u) \geq 0, \qquad \forall \, \tilde{u} \in U_{ad}. \tag{2.13}$$

In [20], it has proved the expression of the control. In this paper, we use the similar method to derive the results below.

Lemma 2.2. *Let u be the solution of (2.9)-(2.13). Then we have*

$$u(x) = \max \left\{ 0, \frac{\bar{z}}{\nu} \right\} - \frac{z}{\nu},$$

where

$$\bar{z} = \frac{\int_{\Omega} z}{\int_{\Omega} 1},$$

denotes the integral average on Ω of the function z.

Proof. For any function $z \in W$, we show that

$$u(x) = \max\left\{0, \frac{\bar{z}}{\nu}\right\} - \frac{z}{\nu},$$

satisfies the variational inequality:

$$(z + \nu u, \tilde{u} - u) \geq 0, \qquad \forall\, \tilde{u} \in U_{ad}.$$

If $\bar{z}/\nu > 0$, then

$$u = \frac{\bar{z}}{\nu} - \frac{z}{\nu},$$

and

$$
\begin{aligned}
&(z + \nu u, \tilde{u} - u) \\
&= \int_\Omega (z + \nu u)(\tilde{u} - u) \\
&= \int_\Omega \bar{z}\left(\tilde{u} - \frac{\bar{z}}{\nu} + \frac{z}{\nu}\right) = \bar{z}\int_\Omega \tilde{u} \geq 0, \qquad \forall\, \tilde{u} \in U_{ad}.
\end{aligned}
\tag{2.14}
$$

If $\bar{z}/\nu \leq 0$, then

$$u = -\frac{z}{\nu}, \quad \text{and} \quad (z + \nu u, \tilde{u} - u) = 0.$$

Note that for the co-state solution z the solution of

$$(z + \nu u, \tilde{u} - u) \geq 0,$$

is unique. Then the lemma is proved.   □

Thus, from above optimality condition (2.13), we have that

$$u(x) = \max\left\{0, \frac{\bar{z}}{\nu}\right\} - \frac{z}{\nu}, \tag{2.15}$$

where

$$\bar{z} = \frac{\int_\Omega z}{\int_\Omega 1},$$

denotes the integral average on Ω of the function z. From the regularity assumption (2.5) and (2.15), we know $u \in H^2(\Omega)$.

Let $\mathcal{T}_h$ denote a regular rectangular partition of the domain Ω, $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the order k Raviart-Thomas mixed finite element space [23]. To approximation the control, we use the following space of piecewise constant functions:

$$U_h = \left\{\tilde{u}_h \in U_{ad} : \quad \tilde{u}_h = \text{constant}, \quad T \in \mathcal{T}_h\right\}. \tag{2.16}$$

Then we introduce the following Raviart-Thomas projection (see [10]):

$$\Pi_h \times P_h : \boldsymbol{V} \times W \longrightarrow \boldsymbol{V}_h \times W_h,$$

which has the following properties:

(i) P_h is the local $L^2(\Omega)$ projection.
(ii) Π_h and P_h satisfy

$$\text{div} \circ \Pi_h = P_h \circ \text{div}, \tag{2.17}$$

$$\left(\text{div}(v - \Pi_h v), w_h\right) = 0, \qquad w_h \in W_h, \tag{2.18}$$

$$(\text{div} v_h, w - P_h w) = 0, \qquad v_h \in V_h. \tag{2.19}$$

(iii) The following approximation properties hold (see [19]):

$$\|v - \Pi_h v\|_{0,\rho} \leq Ch^r \|v\|_{r,\rho}, \qquad \frac{1}{\rho} < r \leq k+1, \tag{2.20}$$

$$\|\text{div}(v - \Pi_h v)\|_{-t} \leq Ch^{r+t} \|\text{div} v\|_r, \qquad 0 \leq r, t \leq k+1, \tag{2.21}$$

$$\|w - P_h w\|_{-t,\rho} \leq Ch^{r+t} \|w\|_{r,\rho}, \qquad 0 \leq r, t \leq k+1, \tag{2.22}$$

where $\|.\|_{r,\rho}$ denotes the norm of the usual Sobolev space $W^{r,\rho}(\Omega)$ for $1 \leq \rho \leq +\infty$ and $r \geq 0$.

The mixed finite element approximation of (2.6)-(2.8) is to find $(p_h, y_h, u_h) \in V_h \times W_h \times U_h$ such that

$$\min_{u \in U_h} \left\{ \frac{1}{2} \| p_h - p_d \|^2 + \frac{1}{2} \|y_h - y_d\|^2 + \frac{\nu}{2} \|u_h\|^2 \right\}, \tag{2.23}$$

$$(A^{-1} p_h, v_h) - (y_h, \text{div} v_h) = 0, \qquad \forall\, v_h \in V_h, \tag{2.24}$$

$$(\text{div} p_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h), \qquad \forall\, w_h \in W_h. \tag{2.25}$$

The control problem (2.23)-(2.25) again has a unique solution (p_h, y_h, u_h), and a triplet $(p_h, y_h, u_h) \in V_h \times W_h \times U_h$ is the solution of (2.23)-(2.25) if and only if there is a co-state $(q_h, z_h) \in V_h \times W_h$ such that $(p_h, y_h, q_h, z_h, u_h)$ satisfies the following discretized optimality conditions:

$$(A^{-1} p_h, v_h) - (y_h, \text{div} v_h) = 0, \qquad \forall\, v_h \in V_h, \tag{2.26}$$

$$(\text{div} p_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h), \qquad \forall\, w_h \in W_h, \tag{2.27}$$

$$(A^{-1} q_h, v_h) - (z_h, \text{div} v_h) = -(p_h - p_d, v_h), \quad \forall\, v_h \in V_h, \tag{2.28}$$

$$(\text{div} q_h, w_h) + (a_0 z_h, w_h) = (y_h - y_d, w_h), \qquad \forall\, w_h \in W_h, \tag{2.29}$$

$$(z_h + \nu u_h, \tilde{u}_h - u_h) \geq 0, \qquad \forall\, \tilde{u}_h \in U_h. \tag{2.30}$$

We shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we define the state solution $\left(p(\tilde{u}), y(\tilde{u}), q(\tilde{u}), z(\tilde{u})\right)$ associated with $\tilde{u}$ which satisfies

$$\left(A^{-1} p(\tilde{u}), v\right) - \left(y(\tilde{u}), \text{div} v\right) = 0, \qquad \forall\, v \in V, \tag{2.31}$$

$$\left(\text{div} p(\tilde{u}), w\right) + \left(a_0 y(\tilde{u}), w\right) = (\tilde{u}, w), \qquad \forall\, w \in W, \tag{2.32}$$

$$\left(A^{-1} q(\tilde{u}), v\right) - \left(z(\tilde{u}), \text{div} v\right) = -\left(p(\tilde{u}) - p_d, v\right), \qquad \forall\, v \in V, \tag{2.33}$$

$$\left(\text{div} q(\tilde{u}), w\right) + \left(a_0 z(\tilde{u}), w\right) = \left(y(\tilde{u}) - y_d, w\right), \qquad \forall\, w \in W. \tag{2.34}$$

Then, we define the discrete state solution $\left(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})\right)$ corresponding to $\tilde{u}$ which satisfies

$$
\begin{align}
&\left(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h\right) - \left(y_h(\tilde{u}), \text{div}\boldsymbol{v}_h\right) = 0, && \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.35) \\
&\left(\text{div}\boldsymbol{p}_h(\tilde{u}), w_h\right) + \left(a_0 y_h(\tilde{u}), w_h\right) = (\tilde{u}, w_h), && \forall\, w_h \in W_h, && (2.36) \\
&\left(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h\right) - \left(z_h(\tilde{u}), \text{div}\boldsymbol{v}_h\right) = -\left(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h\right), && \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, && (2.37) \\
&\left(\text{div}\boldsymbol{q}_h(\tilde{u}), w_h\right) + \left(a_0 z_h(\tilde{u}), w_h\right) = \left(y_h(\tilde{u}) - y_d, w_h\right), && \forall\, w_h \in W_h. && (2.38)
\end{align}
$$

With these definitions, the exact state solution and its corresponding approximations can be written as:

$$
\begin{align*}
(\boldsymbol{p}, y, \boldsymbol{q}, z) &= \left(\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)\right), \\
(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) &= \left(\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)\right).
\end{align*}
$$

3 L^2 superconvergence on rectangular mixed finite elements

Let $\mathcal{T}_h = \{T_i\}$ be a rectangular partition of Ω, $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the lowest order Raviart-Thomas mixed element space, namely,

$$
\begin{align*}
\boldsymbol{V}_h &= \left\{\boldsymbol{v} \in \boldsymbol{V} : \forall\, T_i \in \mathcal{T}_h,\, \boldsymbol{v}\,|_{T_i} \in Q_{1,0}(T_i) \times Q_{0,1}(T_i)\right\}, \\
W_h &= \left\{w \in W : \forall\, T_i \in \mathcal{T}_h,\, w\,|_{T_i} \in Q_{0,0}(T_i)\right\},
\end{align*}
$$

where $Q_{m,n}(T_i)$ indicates the space of polynomials of degree no more than m and n in x and y on T_i, respectively. Thus, on each rectangle element $T_i \in \mathcal{T}_h$, the Gauss point is its center point S_i. For example,

$$
\text{If }\ T_i = [a_i, b_i] \times [c_i, d_i],\quad \text{then } S_i = \left(\frac{a_i + b_i}{2}, \frac{c_i + d_i}{2}\right).
$$

As in [21], for any smooth function $f(x) \in C(\Omega)$, we define an interpolation function f_τ in the following form:

$$
f_\tau(x) = f(S_i), \qquad \text{if } x \in T_i, \tag{3.1}
$$

where S_i is the center point of the rectangle T_i. Let f be a function belonging to $H^2(T_i)$ for all i. Then, by the Bramble-Hilbert Lemma [1], we have

$$
\left|\int_{T_i} (f(x) - f(S_i)) dx\right| \leq Ch^2 \sqrt{|T_i|}\, |f|_{H^2(T_i)}, \tag{3.2}
$$

$$
\sum_i \left|\int_{T_i} (f(x) - f(S_i)) dx\right| \leq Ch^2 \left(\sum_i |f|^2_{H^2(T_i)}\right)^{\frac{1}{2}}, \tag{3.3}
$$

where $|T_i| = \text{meas}(T_i)$.

Let u be the optimal control solution of (1.1)-(1.3), we define a interpolation function $u_I \in U_h$:

$$u_I(x) = \max\left\{0, \overline{\frac{z(S_i)}{\nu}}\right\} - \frac{z(S_i)}{\nu}, \quad \text{if } x \in T_i. \tag{3.4}$$

It is easy to verify that $u_I \in U_h$.

Let z be a function belonging to $H^2(T_i)$ for all i. Then, by (2.15) and the Bramble-Hilbert Lemma [1], we have

$$\left| \int_{T_i} (u(x) - u_I) dx \right| \leq Ch^2 \sqrt{|T_i|} \, |z|_{H^2(T_i)}, \tag{3.5}$$

$$\sum_i \left| \int_{T_i} (u(x) - u_I) dx \right| \leq Ch^2 \left(\sum_i |z|_{H^2(T_i)}^2 \right)^{\frac{1}{2}}. \tag{3.6}$$

Before presenting the main theorem, we first give some useful lemmas that have been proved in [6] and derive the main lemma of the section.

Lemma 3.1. *Suppose that Assumptions (**A1**)-(**A2**) are valid. Let $z_h(u)$ and $z_h(u_h)$ be the discrete solutions of (2.35)-(2.38) with $\tilde{u} = u$ and $\tilde{u} = u_h$, respectively. Then we have*

$$(z_h(u_h) - z_h(u_I), u_I - u_h) \leq 0. \tag{3.7}$$

Lemma 3.2. *Suppose that Assumptions (**A1**)-(**A2**) are valid. For any function $\tilde{u} \in U_{ad}$, let $(p(\tilde{u}), y(\tilde{u}), q(\tilde{u}), z(\tilde{u}))$ and $(p_h(\tilde{u}), y_h(\tilde{u}), q_h(\tilde{u}), z_h(\tilde{u}))$ be the solutions of (2.31)-(2.34) and (2.35)-(2.38), respectively, for the lowest order Raviart-Thomas mixed finite elements. If the regularity conditions*

$$y(\tilde{u}), \, z(\tilde{u}) \in H^1(\Omega), \quad p(\tilde{u}), \, q(\tilde{u}) \in \left(H^2(\Omega)\right)^2,$$

hold, then we have

$$\|P_h y(\tilde{u}) - y_h(\tilde{u})\| + \|\Pi_h p(\tilde{u}) - p_h(\tilde{u})\| \leq Ch^2 \left(\|p(\tilde{u})\|_{H^2(\Omega)} + \|y(\tilde{u})\|_{H^1(\Omega)} \right), \tag{3.8}$$

$$\|P_h z(\tilde{u}) - z_h(\tilde{u})\| + \|\Pi_h q(\tilde{u}) - q_h(\tilde{u})\| \leq Ch^2 \left(\|q(\tilde{u})\|_{H^2(\Omega)} + \|z(\tilde{u})\|_{H^1(\Omega)} \right). \tag{3.9}$$

Lemma 3.3. *Suppose that Assumptions (**A1**)-(**A2**) are fulfilled. Let $z(u_I)$ and $z_h(u_I)$ be the solutions of (2.31)-(2.34) and (2.35)-(2.38) with $\tilde{u} = u_I$, respectively. Then we have*

$$(z_h(u_I) - z(u_I), u_I - u_h) \leq Ch^{1+s} \|z\|_{H^{2+s}(\Omega)} \cdot \|u_I - u_h\|, \quad 0 < s \leq 1. \tag{3.10}$$

Lemma 3.4. *Suppose that Assumptions (**A1**)-(**A2**) are fulfilled. Let u_h be the solution of (2.26)-(2.30) and z_τ and u_I be the interpolation functions of z and u defined in (3.1) and (3.4), respectively. Then we have*

$$(z - z_\tau, u_I - u_h) \leq Ch^2 \|z\|_{H^2(\Omega)} \cdot \|u_I - u_h\|. \tag{3.11}$$

Now, we prove the main lemma of the section.

Lemma 3.5. *Suppose that Assumptions* (**A1**)-(**A2**) *are valid. Let u_I be the interpolation of the exact control u defined in (3.1) and $z(u_I)$ and $z(u)$ be the solutions of (2.31)-(2.34) with $\tilde{u} = u_I$ and $\tilde{u} = u$, respectively. Then we have*

$$\|z(u_I) - z(u)\| \le Ch^2. \tag{3.12}$$

Proof. From the Assumption (**A2**), we use the strong forms of the Eqs. (1.2) and (2.1) to obtain the error equations:

$$-\text{div}\Big(A(x)\mathbf{grad}\big(y(u_I) - y(u)\big)\Big) + a_0\big(y(u_I) - y(u)\big) = u_I - u, \tag{3.13}$$

$$-\text{div}\Big(A(x)\mathbf{grad}\big(z(u_I) - z(u)\big) + \boldsymbol{p}(u_I) - \boldsymbol{p}(u)\Big)$$
$$+a_0\Big(z(u_I) - z(u)\Big) = y(u_I) - y(u), \tag{3.14}$$

which imply that

$$\|z(u_I) - z(u)\|_{H^1(\Omega)} \le C\Big(\|y(u_I) - y(u)\| + \|\boldsymbol{p}(u_I) - \boldsymbol{p}(u)\|\Big). \tag{3.15}$$

Then, we multiply (3.14) by $y(u_I) - y(u)$ to derive that

$$\|y(u_I) - y(u)\|^2$$
$$=\big(y(u_I) - y(u), y(u_I) - y(u)\big)$$
$$=-\Big(\text{div}\big(A(x)\mathbf{grad}\big(z(u_I) - z(u)\big) + \boldsymbol{p}(u_I) - \boldsymbol{p}(u)\big), y(u_I) - y(u)\Big)$$
$$+\Big(a_0\big(z(u_I) - z(u)\big), y(u_I) - y(u)\Big)$$
$$=\Big(A(x)\mathbf{grad}\big(z(u_I) - z(u)\big), \mathbf{grad}\big(y(u_I) - y(u)\big)\Big)$$
$$+\Big(A(x)\big(\boldsymbol{p}(u_I) - \boldsymbol{p}(u)\big), \mathbf{grad}\big(y(u_I) - y(u)\big)\Big) + \Big(a_0\big(z(u_I) - z(u)\big), y(u_I) - y(u)\Big)$$
$$=\Big(A(x)\mathbf{grad}\big(y(u_I) - y(u)\big), \mathbf{grad}\big(z(u_I) - z(u)\big)\Big)$$
$$-\Big(\boldsymbol{p}(u_I) - \boldsymbol{p}(u), \boldsymbol{p}(u_I) - \boldsymbol{p}(u)\Big) + \Big(a_0\big(y(u_I) - y(u)\big), z(u_I) - z(u)\Big)$$
$$=\Big(u_I - u, z(u_I) - z(u)\Big) - \Big(\boldsymbol{p}(u_I) - \boldsymbol{p}(u), \boldsymbol{p}(u_I) - \boldsymbol{p}(u)\Big),$$

where we have used

$$\boldsymbol{p}(u_I) - \boldsymbol{p}(u) = -A(x)\mathbf{grad}\Big(y(u_I) - y(u)\Big),$$

and (3.13). Thus, we have the following identity

$$\|y(u_I) - y(u)\|^2 + \|\boldsymbol{p}(u_I) - \boldsymbol{p}(u)\|^2 = \Big(u_I - u, z(u_I) - z(u)\Big). \tag{3.16}$$

Now, we define a standard piecewise linear function space

$$S_h = \left\{ r_h(x) \in C(\overline{\Omega}) : \ r_h \in Q_{1,1}(T_i), \ \forall \ T_i \in \mathcal{T}_h \right\}, \tag{3.17}$$

and a standard $H^1(\Omega)$-orthogonal projection $Q_h : C(\overline{\Omega}) \to S_h$, which satisfies: for any $\psi \in C(\overline{\Omega})$

$$\left(\mathbf{grad}(\psi - Q_h\psi), \mathbf{grad} r_h \right) = 0, \quad \forall \ r_h \in S_h. \tag{3.18}$$

By standard finite element analysis, the projection Q_h has the following approximate property and stable property:

$$\|\psi - Q_h\psi\| \le Ch\|\psi\|_{H^1(\Omega)}, \tag{3.19}$$

$$\|Q_h\psi\|_{H^1(\Omega)} \le C\|\psi\|_{H^1(\Omega)}. \tag{3.20}$$

Set $r = z(u_I) - z(u)$ and $r_h = Q_h\big(z(u_I) - z(u)\big)$. We can write the right-hand side of Eq. (3.16) as follows

$$\left(u_I - u, z(u_I) - z(u) \right) = (u_I - u, r - r_h) + (u_I - u, r_h). \tag{3.21}$$

It follows from (3.19) that

$$(u_I - u, r - r_h) \le \|u_I - u\| \cdot \|r - r_h\|$$
$$\le Ch^2\|u\|_{W^{1,\infty}} \cdot \|z(u_I) - z(u)\|_{H^1(\Omega)}. \tag{3.22}$$

Since $r_h \in Q_{1,1}(T_i)$ for any rectangle T_i, then we have

$$\int_{T_i} u_I r_h dx = \int_{T_i} u_I(S_i) r_h dx = \int_{T_i} u_I(S_i) r_h(S_i) dx. \tag{3.23}$$

By using (3.23) and (3.2), we obtain that

$$\left| \int_{\Omega} (u_I - u) r_h \right|$$
$$\le \sum_{T_i \in \Omega} \left| \int_{T_i} \big(u(S_i) - u(x) \big) \cdot r_h(x) dx \right|$$
$$= \sum_{T_i \in \Omega} \left| \int_{T_i} \big[(u r_h)(S_i) - (u r_h)(x) \big] dx \right|$$
$$\le Ch^2 \left(\sum_{T_i \in \Omega} |u r_h|^2_{H^2(T_i)} \right)^{\frac{1}{2}} \le Ch^2 \left(\sum_{T_i \in \Omega} |u|^2_{H^2(T_i)} \cdot |r_h|^2_{H^1(T_i)} \right)^{\frac{1}{2}}.$$

From (2.15), we obtain that

$$|u|_{H^2(T_i)} \le C|z|_{H^2(T_i)}. \tag{3.24}$$

Applying the estimate in H^1 norm for L^2 projections (see [5,14]), we have

$$|r_h|_{H^1(T_i)} = |Q_h(z(u_I) - z(u))|_{H^1(T_i)} \le C|z(u_I) - z(u)|_{H^1(T_i)}.$$

Therefore,

$$
\left| \int_\Omega (u_I - u)r_h \right| \le Ch^2 \left(\sum_{T_i \in \Omega} |z|^2_{H^2(T_i)} \cdot |z(u_I) - z(u)|^2_{H^1(T_i)} \right)^{\frac{1}{2}}
$$
$$
\le Ch^2 \cdot \|z\|_{H^2(\Omega)} \cdot |z(u_I) - z(u)|_{H^1(\Omega)}. \tag{3.25}
$$

Finally, we can combine the estimates (3.16) and (3.21)-(3.25) to derive

$$
\|y(u_I) - y(u)\|^2 + \|p(u_I) - p(u)\|^2
$$
$$
\le Ch^2 \cdot \|z(u_I) - z(u)\|_{H^1(\Omega)}
$$
$$
\le Ch^2 \cdot (\|y(u_I) - y(u)\| + \|p(u_I) - p(u)\|),
$$

where we have used the stability property (3.15) in the last step. The above inequality implies the desired result (3.12). $\square$

Now, we are able to obtain our first main result.

Theorem 3.1. *Suppose that Assumptions* (**A1**)-(**A2**) *are satisfied. Let u_I be the interpolation of the exact control u defined in* (3.4) *and u_h be the solution of* (2.26)-(2.30). *Then we have the estimate*

$$\|u_I - u_h\| \le Ch^{1+s}, \qquad 0 < s \le 1. \tag{3.26}$$

Proof. From the inequality (2.13), we have

$$\Big(z(x) + \nu u(x)\Big) \cdot \Big(\tilde{u} - u(x)\Big) \ge 0, \qquad \forall\, \tilde{u} \in U_{ad}, \quad \forall\, x \in \Omega. \tag{3.27}$$

We apply this formula for $x = S_i$, and $\tilde{u} = u_h(S_i)$. This is correct because of the continuity of u, z, and u_h in these points $\{S_i\}$, namely,

$$\Big(z(S_i) + \nu u(S_i)\Big) \cdot \Big(u_h(S_i) - u(S_i)\Big) \ge 0, \qquad \forall\, S_i. \tag{3.28}$$

Due to (3.1), the above inequality is equivalent to

$$\Big(z(S_i) + \nu u_I(S_i)\Big) \cdot \Big(u_h(S_i) - u_I(S_i)\Big) \ge 0, \qquad \forall\, S_i. \tag{3.29}$$

Integrating this inequality over T_i and adding up over all i, we get that

$$(z_\tau + \nu u_I, u_h - u_I) \ge 0. \tag{3.30}$$

We choose the test function $\tilde{u}_h = u_I$ in (2.30) to obtain that

$$\Big(z_h(u_h) + \nu u_h, u_I - u_h\Big) \ge 0. \tag{3.31}$$

By adding these two inequalities (3.30) and (3.31), we have

$$\Big(z_h(u_h) - z_\tau + \nu(u_h - u_I), u_I - u_h\Big) \geq 0.$$

Hence,

$$\begin{aligned}
&\nu\|u_I - u_h\|^2 \\
&\leq \Big(z_h(u_h) - z_\tau, u_I - u_h\Big) \\
&= \Big(z_h(u_h) - z_h(u_I), u_I - u_h\Big) + \Big(z_h(u_I) - z(u_I), u_I - u\Big) \\
&\quad + \Big(z(u_I) - z(u), u_I - u_h\Big) + \Big(z - z_\tau, u_I - u_h\Big).
\end{aligned} \tag{3.32}$$

Then we combine Lemma 3.1 and Lemmas 3.3-3.5 to deduce the superconvergence result (3.26). □

Next, we can establish the following superconvergence result for state and co-state.

Theorem 3.2. *Suppose that Assumptions* (**A1**)-(**A2**) *are satisfied. Let* $(p, y, q, z, u) \in (V \times W)^2 \times U_{ad}$ *be the solutions defined in* (2.9)-(2.13) *and* $(p_h, y_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times U_h$ *be the solutions of* (2.26)-(2.30). *Then we have*

$$\|\Pi_h p - p_h\|_{\mathrm{div}} + \|P_h y - y_h\| \leq Ch^{1+\min(s,\frac{1}{2})}, \tag{3.33}$$

$$\|\Pi_h q - q_h\|_{\mathrm{div}} + \|P_h z - z_h\| \leq Ch^{1+\min(s,\frac{1}{2})}, \tag{3.34}$$

for $0 < s \leq 1$.

Proof. It follows from (2.9)-(2.12) and (2.26)-(2.30) that we have the error equations:

$$\begin{aligned}
&\big(A^{-1}(p - p_h), v_h\big) - (y - y_h, \mathrm{div}v_h) = 0, \\
&(\mathrm{div}(p - p_h), w_h) + \big(a_0(y - y_h), w_h\big) = (u - u_h, w_h), \\
&\big(A^{-1}(q - q_h), v_h\big) - (z - z_h, \mathrm{div}v_h) = -(p - p_h, v_h), \\
&(\mathrm{div}(q - q_h), w_h) + \big(a_0(z - z_h), w_h\big) = (y - y_h, w_h),
\end{aligned}$$

for all $v_h \in V_h$ and $w_h \in W_h$. By using the definitions of projections Π_h and P_h, the above equations can be rewrittrn as

$$\begin{aligned}
&\big(A^{-1}(\Pi_h p - p_h), v_h\big) - (P_h y - y_h, \mathrm{div}v_h) = \phi_1(v_h), \\
&(\mathrm{div}(\Pi_h p - p_h), w_h) + \big(a_0(P_h y - y_h), w_h\big) = \psi_1(w_h), \\
&\big(A^{-1}(\Pi_h q - q_h), v_h\big) - (P_h z - z_h, \mathrm{div}v_h) = \phi_2(v_h), \\
&(\mathrm{div}(\Pi_h q - q_h), w_h) + \big(a_0(P_h z - z_h), w_h\big) = \psi_2(w_h),
\end{aligned}$$

for all $v_h \in V_h$ and $w_h \in W_h$, where

$$\begin{aligned}
\phi_1(v_h) &= -\big(A^{-1}(p - \Pi_h p), v_h\big), \\
\psi_1(w_h) &= (u - u_h, w_h) - \big(a_0(y - P_h y), w_h\big), \\
\phi_2(v_h) &= -(p - p_h, v_h) - \big(A^{-1}(q - \Pi_h q), v_h\big), \\
\psi_2(w_h) &= (y - y_h, w_h) - \big(a_0(z - P_h z), w_h\big).
\end{aligned}$$

Since the terms $\phi_1(v_h), \psi_1(w_h), \phi_2(v_h), \psi_2(w_h)$ can be regarded as linear functionals of v_h and w_h defined on V_h and W_h, respectively, we know from the stability result of [4, 22] that

$$\|\Pi_h p - p_h\|_{\mathrm{div}} + \|P_h y - y_h\| \leq C \left\{ \sup_{v_h \in V_h} \frac{|\phi_1(v_h)|}{\|v_h\|_{\mathrm{div}}} + \sup_{w_h \in W_h} \frac{|\psi_1(w_h)|}{\|w_h\|} \right\}, \tag{3.35}$$

$$\|\Pi_h q - q_h\|_{\mathrm{div}} + \|P_h z - z_h\| \leq C \left\{ \sup_{v_h \in V_h} \frac{|\phi_2(v_h)|}{\|v_h\|_{\mathrm{div}}} + \sup_{w_h \in W_h} \frac{|\psi_2(w_h)|}{\|w_h\|} \right\}. \tag{3.36}$$

It is easy to see that

$$(p - p_h, v_h) = (p - \Pi_h p, v_h) + (\Pi_h p - p_h, v_h), \tag{3.37}$$
$$(y - y_h, w_h) = (y - P_h y, w_h) + (P_h y - y_h, w_h) = (P_h y - y_h, w_h). \tag{3.38}$$

By the standard superconvergence of mixed finite element methods, we have

$$(a_0(y - P_h y), w_h) \leq Ch^2 \|y\|_{H^1(\Omega)} \|w_h\|, \tag{3.39}$$
$$(a_0(z - P_h z), w_h) \leq Ch^2 \|z\|_{H^1(\Omega)} \|w_h\|. \tag{3.40}$$

Under the condition $y, z \in H^3(\Omega)$, applying the integral identity technique [16] gives

$$(A^{-1}(p - \Pi_h p), v_h) \leq Ch^2 \|y\|_{H^3(\Omega)} \|v_h\|, \tag{3.41}$$
$$(A^{-1}(q - \Pi_h q), v_h) \leq Ch^2 \|z\|_{H^3(\Omega)} \|v_h\|, \tag{3.42}$$
$$(p - \Pi_h p, v_h) \leq Ch^2 \|y\|_{H^3(\Omega)} \|v_h\|. \tag{3.43}$$

On the other hand, applying the standard error estimates of mixed finite element methods and the approximation properties of projection operators P_h and Π_h, we have that

$$(A^{-1}(p - \Pi_h p), v_h) \leq Ch \|y\|_{H^2(\Omega)} \|v_h\|, \tag{3.44}$$
$$(A^{-1}(q - \Pi_h q), v_h) \leq Ch \|z\|_{H^2(\Omega)} \|v_h\|, \tag{3.45}$$
$$(p - \Pi_h p, v_h) \leq Ch \|y\|_{H^2(\Omega)} \|v_h\|. \tag{3.46}$$

Then, by the interpolation theory, under the assumption (**A2**) we obtain that

$$(A^{-1}(p - \Pi_h p), v_h) \leq Ch^{1+s} \|y\|_{H^{2+s}(\Omega)} \|v_h\|, \tag{3.47}$$
$$(A^{-1}(q - \Pi_h q), v_h) \leq Ch^{1+s} \|z\|_{H^{2+s}(\Omega)} \|v_h\|, \tag{3.48}$$
$$(p - \Pi_h p, v_h) \leq Ch^{1+s} \|y\|_{H^{2+s}(\Omega)} \|v_h\|. \tag{3.49}$$

Here, we only give the proof of (3.47). We define a linear functional

$$Ty = (A^{-1}(p - \Pi_h p), v_h) = \left(A^{-1}(-A\,\mathbf{grad}y - \Pi_h(-A\mathbf{grad}y)), v_h \right).$$

Then, it follows from (3.41) and (3.44) that

$$\|T\|_{L(H^{2+s}(\Omega)\to\mathbf{R})} \leq \|T\|_{L(H^3(\Omega)\to\mathbf{R})}^s \|T\|_{L(H^2(\Omega)\to\mathbf{R})}^{1-s}$$
$$\leq C(h^2\|v_h\|)^s \cdot (h\|v_h\|)^{1-s} = Ch^{1+s}\|v_h\|,$$

which implies (3.47). We can similarly prove (3.48) and (3.49). Note that

$$(u - u_h, w_h) = (u - u_I, w_h) + (u_I - u_h, w_h). \tag{3.50}$$

It follows from (3.1)-(3.3) that

$$(u - u_I, w_h) \leq \sum_{T_i \in \Omega} |w_h| \left| \int_{T_i} (u(x) - u(S_i)) dx \right|$$
$$\leq Ch^2 \sum_{T_i \in \Omega} \|u\|_{H^2(T_i)} \cdot \sqrt{|T_i|} \cdot |w_h| = Ch^2 \|u\|_{H^2(\Omega)} \cdot \|w_h\|.$$

By using Theorem 3.1, we clearly see that

$$(u_I - u_h, w_h) \leq \|u_I - u_h\| \cdot \|w_h\| \leq Ch^{1+s}\|w_h\|. \tag{3.51}$$

From the above analysis, we can obtain the desired results (3.33)-(3.34). □

4   Global L^2 superconvergence by postprocessing

In this section, we shall apply a higher order interpolation postprocessing method presented by Lin and Yan [16] to obtain global superconvergence for the approximation. We construct a larger rectangular elements partition $\mathcal{T}_{2h}$, which is the coarse meshes of $\mathcal{T}_h$. Then each element τ of $\mathcal{T}_{2h}$ is composed of four neighboring rectangular elements of $\mathcal{T}_h$. Based on this coarse meshes, we denote $V_{2h} \times W_{2h}$ to express the order $k = 1$ Raviart-Thomas mixed finite element spaces:

$$V_{2h} = \left\{ v \in V : \forall \tau \in \mathcal{T}_{2h}, \ v\mid_\tau \in Q_{2,1}(\tau) \times Q_{1,2}(\tau) \right\},$$
$$W_{2h} = \left\{ w \in W : \forall \tau \in \mathcal{T}_{2h}, \ w\mid_\tau \in Q_{1,1}(\tau) \right\},$$

and the related Raviart-Thomas projection (see [10]):

$$\Pi_{2h} \times P_{2h} : \ V \times W \to V_{2h} \times W_{2h},$$

which satisfies the following properties [23]:

(i) $P_{2h}P_h = P_{2h}$ and $\|P_{2h}w_h\| \leq C\|w_h\|$, for all $w_h \in W_h$.

(ii) $\Pi_{2h}\Pi_h = \Pi_{2h}$ and $\|\Pi_{2h}v_h\|_{\mathrm{div}} \leq C\|v_h\|_{\mathrm{div}}$, for all $v_h \in V_h$.

By using the interpolation operators Π_{2h} and P_{2h} and their properties, we can obtain the following global superconvergence result.

Lemma 4.1. ([6]) *For any function $\tilde{u} \in U_{ad}$, let*

$$\left(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u})\right), \quad \text{and} \quad \left(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u})\right),$$

be the solutions of (2.31)-(2.34) and (2.35)-(2.38), respectively, with the lowest order Raviart-Thomas mixed finite elements. If the regularity conditions

$$y(\tilde{u}), \quad z(\tilde{u}) \in H^1(\Omega), \quad \boldsymbol{p}(\tilde{u}), \quad \boldsymbol{q}(\tilde{u}) \in \left(H^2(\Omega)\right)^2,$$

hold, then we have

$$\|p(\tilde{u}) - P_{2h}p_h(\tilde{u})\| + \|\boldsymbol{p}(\tilde{u}) - \Pi_{2h}\boldsymbol{p}_h(\tilde{u})\| \leq Ch^2,$$
$$\|z(\tilde{u}) - P_{2h}z_h(\tilde{u})\| + \|\boldsymbol{q}(\tilde{u}) - \Pi_{2h}\boldsymbol{q}_h(\tilde{u})\| \leq Ch^2.$$

In order to improve the accuracy of the control approximation on a global scale, we construct

$$\hat{u}(x) = \max\left\{0, \frac{\bar{z}}{\nu}\right\} - \frac{P_{2h}z_h}{\nu}. \tag{4.1}$$

Now, we can prove the following global L^2 superconvergence result.

Theorem 4.1. *Suppose that Assumptions* (**A1**)-(**A2**) *are satisfied. Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ *be the solutions defined in (2.9)-(2.13) and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the solutions of (2.26)-(2.30). Then we have*

$$\|u - \hat{u}\| \leq Ch^{1+s}, \qquad 0 < s \leq 1. \tag{4.2}$$

Proof. From (2.15) and (4.1), we obtain by the triangle inequality

$$\|u - \hat{u}\| \leq C\|z - P_{2h}z_h\|$$
$$\leq C\Big(\|z(u) - z(u_I)\| + \|z(u_I) - P_{2h}z_h(u_I)\| + \|P_{2h}z_h(u_I) - P_{2h}z_h(u_h)\|\Big). \tag{4.3}$$

We first apply lemma 3.5 to obtain that

$$\|z(u) - z(u_I)\| \leq Ch^2. \tag{4.4}$$

Then, from the approximation property of the operator P_{2h}, the property (i) of the operator P_{2h}, we have

$$\begin{aligned}
&\|z(u_I) - P_{2h}z_h(u_I)\| \\
\leq& \|z(u_I) - P_{2h}z(u_I)\| + \|P_{2h}z(u_I) - P_{2h}z_h(u_I)\| \\
\leq& Ch^2\|z(u_I)\|_{H^2(\Omega)} + \|P_{2h}P_hz(u_I) - P_{2h}z_h(u_I)\| \\
\leq& Ch^2\|z(u_I)\|_{H^2(\Omega)} + \|P_hz(u_I) - z_h(u_I)\| \\
\leq& Ch^{1+s}\|z\|_{H^{2+s}(\Omega)}.
\end{aligned} \tag{4.5}$$

Next, it remains to bound the third term of above inequality (4.3). By the property (i) of the operator P_{2h}, we have

$$\|P_{2h}z_h(u_I) - P_{2h}z_h(u_h)\| \leq C\|z_h(u_I) - z_h(u_h)\|.$$

Similar to the proof of Lemma 3.1, we use (2.35)-(2.38) to obtain the error equations

$$\left(A^{-1}(\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h)), \boldsymbol{v}_h\right) - \left(y_h(u_I) - y_h(u_h), \mathrm{div}\boldsymbol{v}_h\right) = 0,$$

$$\left(\mathrm{div}(\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h)), w_h\right) + \left(a_0(y_h(u_I) - y_h(u_h)), w_h\right) = (u_I - u_h, w_h),$$

$$\left(A^{-1}(\boldsymbol{q}_h(u_I) - \boldsymbol{q}_h(u_h)), \boldsymbol{v}_h\right) - \left(z_h(u_I) - z_h(u_h), \mathrm{div}\boldsymbol{v}_h\right) = -\left(\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h), \boldsymbol{v}_h\right),$$

$$\left(\mathrm{div}(\boldsymbol{q}_h(u_I) - \boldsymbol{q}_h(u_h)), w_h\right) + \left(a_0(z_h(u_I) - z_h(u_h)), w_h\right) = (y_h(u_I) - y_h(u_h), w_h),$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$. We use the stability property of the saddle-point problem to obtain that

$$\begin{aligned}
&\|z_h(u_I - z_h(u_h)\| + \|\boldsymbol{q}_h(u_I) - \boldsymbol{q}_h(u_h)\| \\
\leq &C\left(\|y_h(u_I) - y_h(u_h)\| + \|\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h)\|\right) \\
\leq &C\|u_I - u_h\| \leq Ch^{1+s},
\end{aligned} \tag{4.6}$$

where the last step was derived by using Theorem 3.1. Then by (4.3)-(4.6), we can prove the result (4.2). □

5 Numerical tests

In this section, we present below two examples to test the superconvergence theoretical results of the control. The first example is based on Example 1 of [6] with some modification. In the second example, we consider the control problem with a nonlinear state equation.

The optimization problems were solved numerical by projected gradient methods, with codes developed based on AFEPACK [13]. The control function u is discretized by piecewise constant functions, where the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximated by the lowest order Raviart-Thomas mixed finite element functions. In the two examples, we choose the domain $\Omega = [0, 1] \times [0, 1]$.

Example 1. We consider the following two-dimensional elliptic optimal control problem

$$\min_{u \in U_{ad}} \frac{1}{2} \left\{ \|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right\}, \tag{5.1}$$

subject to the state equation

$$\mathrm{div}\boldsymbol{p} + a_0 y = u + f, \quad \boldsymbol{p} = -\mathbf{grad}y, \qquad x \in \Omega, \tag{5.2}$$

with the boundary condition

$$y = 0, \qquad x \in \partial\Omega, \tag{5.3}$$

and the admissible set

$$U_{ad} = \left\{ u \in L^2(\Omega) : \int_\Omega u(x)dx \geq 0 \right\}. \tag{5.4}$$

Let $a_0 = 0$. Then the state equation may be restated as

$$\mathrm{div}\,\boldsymbol{p} = u + f, \quad \boldsymbol{p} = -\mathbf{grad}y, \qquad x \in \Omega, \tag{5.5}$$

Next, we introduce the co-state elliptic equation

$$\mathrm{div}\,\boldsymbol{q} = y - y_d, \quad \boldsymbol{q} = -(\mathbf{grad}z + \boldsymbol{p} - \boldsymbol{p}_d), \qquad x \in \Omega, \tag{5.6}$$

with the boundary condition

$$z = 0, \qquad x \in \partial\Omega. \tag{5.7}$$

We choose

$$
\begin{aligned}
&y = \sin(\pi x_1)\sin(\pi x_2), \qquad z = -2\pi^2 \sin(\pi x_1)\sin(\pi x_2), \\
&u = \max(0, \bar{z}) - z, \qquad\qquad f = 2\pi^2 y - u, \\
&\boldsymbol{p}_d = \left(-\pi(1+\pi^2)\cos(\pi x_1)\sin(\pi x_2), -\pi(1+\pi^2)\sin(\pi x_1)\cos(\pi x_2) \right), \\
&y_d = (1 + 2\pi^4)\sin(\pi x_1)\sin(\pi x_2).
\end{aligned}
$$

In the numerical implementation, the profile of the numerical solution is plotted in Fig. 1 and the errors $\|u_I - u_h\|$ and $\|u_I - \hat{u}\|$ obtained on a sequence of uniformly
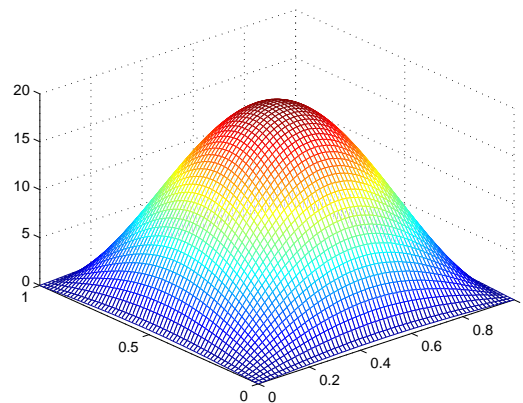

Figure 1: The profile of the numerical solution of Example 1 on an 64×64 mesh grid.

Table 1: The errors of Example 1 on a sequential uniform refined meshes.

resolution	$\|u_I - u_h\|$	$\|u - \hat{u}\|$
16×16	4.506e-02	4.324e-01
32×32	1.129e-02	1.477e-01
64×64	2.831e-03	5.907e-02
128×128	7.149e-04	1.776e-02

refined meshes are presented in Table 1. The superconvergence phenomenon can be observed clearly from the data.

Example 2. In this example, we consider the following nonlinear optimal control problem

$$\min_{u \in U_{ad}} \frac{1}{2} \left\{ \|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right\},$$
$$- \operatorname{div}(\mathbf{grad} y) + y^5 = u + f, \quad x \in \Omega,$$
$$y = 0, \qquad x \in \partial\Omega,$$

and we introduce co-state elliptic equation

$$\operatorname{div}\boldsymbol{q} + 5y^4 z = y - y_d, \quad \boldsymbol{q} = -(\mathbf{grad} z + \boldsymbol{p} - \boldsymbol{p}_d), \qquad x \in \Omega, \tag{5.8}$$

with the boundary condition

$$z = 0, \qquad x \in \partial\Omega.$$

We choose that

$$y = \sin(\pi x_1) \sin(\pi x_2), \qquad z = -\pi^2 \sin(\pi x_1) \sin(\pi x_2),$$
$$u = \max(0, \bar{z}) - z, \qquad f = 2\pi^2 y + y^5 - u,$$
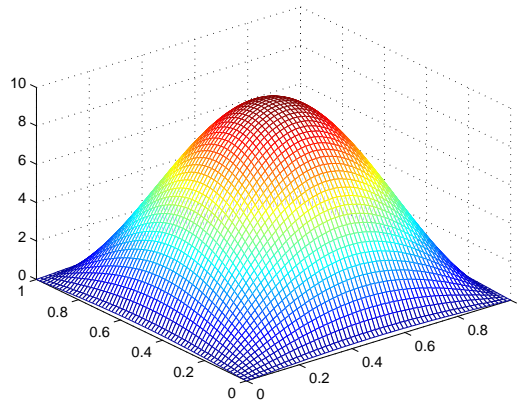

Figure 2: The profile of the numerical solution of Example 2 on 64×64 mesh grids.

Table 2: The errors of Example 2 on a sequential uniform refined meshes.

resolution	$\|u_I - u_h\|$	$\|u - \hat{u}\|$
16×16	1.049e-02	2.118e-01
32×32	2.617e-03	7.296e-02
64×64	6.530e-04	2.531e-02
128×128	1.623e-04	8.850e-03

$$\boldsymbol{p}_d = \boldsymbol{p} + \mathbf{grad}z = \Big(-\pi(1 + \pi^2)\cos(\pi x_1)\sin(\pi x_2),$$
$$-\pi(1 + \pi^2)\sin(\pi x_1)\cos(\pi x_2) \Big),$$
$$y_d = y - 5y^4 z.$$

The profile of the numerical solution is presented in Fig. 2. The superconvergence behavior of the L^2-errors is illustrated in Table 2.

6   Conclusions and future work

In this paper, we have discussed the lowest order Raviart-Thomas mixed finite element methods for constrained quadratic optimal control problem, and the admissible set:

$$U_{ad} = \left\{ u \in L^2(\Omega) : \int_\Omega u(x)dx \geq 0 \right\}.$$

We have obtained the superconvergence of $\mathcal{O}(h^{1+s})$ $(0 < s \leq 1)$ for the control variable which is approximated by piecewise constant functions.

In our future work, we shall use the mixed finite element method to deal with the optimal control problems governed by nonlinear parabolic equations and convex boundary control problems. Furthermore, we shall consider a priori error estimates and superconvergence of optimal control problems governed by nonlinear parabolic equations and convex boundary control problems.

Acknowledgements

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