Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 2, No. 1, pp. 86-106

DOI: 10.4208/aamm.09-m0927
February 2010

Effects of Heat and Mass Transfer on the Perfect Conducting Polar Fluid Over a Stretching Sheet Due to Thermal Radiation

M. Zakaria[1,*]

[1] *Department of Mathematics, Faculty of Science, Al-Baha University, P.O. Box 1988 Al-Baha Kingdom of Saudi Arabia*

Received 21 March 2009; Accepted (in revised version) 26 June 2009
Available online 31 December 2009

Abstract. The objectives of the present study are to investigate the effects of heat and mass transfer on the unsteady flow of a perfect conducting polar fluid over a flat sheet with a linear velocity in the presence of thermal radiation. Successive approximation method developed in Zakaria [Appl. Math. Comput. 10 (2002), Appl. Math. Comput. 155 (2005)] is adopted to solve the problem. The effects of Alfven velocity, mass transfer, various material parameters, Prandtl number, Schmidt number and relaxation time on the velocity, angular velocity, temperature and concentration are discussed and illustrated graphically.

AMS subject classifications: 80A20

Key words: Heat and mass transfer, relaxation time, perfectly conducting, polar fluid.

1 Introduction

Due to the increase of importance in the processing industries and elsewhere of materials whose flow behavior in shear cannot be characterized by classical fluid, a new stage in the evolution of fluid dynamic theory achieves development. Hoyt and Fabula [1], Vogel and Patterson [2] have shown experimentally that fluids containing minute polymeric additives indicate considerable reduction of the skin friction near a rigid body (about 25-30%), a concept which can be well explained by the theory of polar fluids. The classical fluid mechanics could not explain this phenomenon. Therefore, Aero et al. [3] and D'ep [4] have proposed the theory of the polar fluids taking in consideration that the inertial characteristics of the substructure particles which are allowed to undergo rotation. This theory can be used to explain the flow of the colloidal fluids liquid crystals animal blood etc.

*Corresponding author.
Email: zakariandm@yahoo.com (M. Zakaria)

Polar fluid dynamics is concerned with the fluids motion whose material points possess orientations. It is distinguished from classical fluid dynamics (which is also known as Newtonian fluid dynamics or Navier-Stokes (N-S) theory) in the classical fluid dynamics, which is not assumed to possess oriented material points. Thus, against the three translational degrees of freedom of the classical theory, polar fluids possess six degrees of freedom: three of them are translational degrees and the other rotational degrees. The rotational degrees of freedom play a role in the nonsymmetrical stress tensors and couple tresses, which are missing from the classical theory.

Boundary layer flow on continuous moving surfaces is an important type of flow occurring in a number of engineering processes. Aerodynamic extrusion of plastic and rubber sheets, cooling of an infinite metallic plate in a cooling path, which may be an electrolyte, crystal growing, the boundary layer along a liquid film in condensation processes and a polymer sheet or filament extruded continuously from a die, or along thread traveling between a feed roll and a wind-up roll are examples of practical applications of continuous moving surfaces. Glass blowing, continuous casting, and spinning of fibers also involve the flow due to the stretching surface. Various aspects of this problem have been studied by Sakiadis [5–7]. Eldabe and Ouaf [8] studied the heat and mass transfer in a hydromagnetics flow of a micro polar fluid past a stretching surface. Kleson and Desseaux analyzed the effect of surface conditions on flow of a micro polar fluid driven by a porous stretching sheet [9]. Massive amount of works have been done on heat and mass transfer for a hydromagnetics flow over a stretching sheet [10–13].

In the above mentioned works, the effect of the induced magnetic field was neglected. Recently Zakaria [14, 15], also Ezzat and Zakaria [16, 17] studied the effects of induced magnetic field and heat transfer of polar and viscoelastic fluid flow over a stretching sheet. The modification of the heat-conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modifying the classical Fourier law of heat conduction.

In this work, we use a more general model of MHD perfect conducting polar fluid over a stretching sheet due to thermal radiation under the effects of heat and mass transfer, which also includes the relaxation time of concentration, heat conduction and the electric displacement current [18, 19]. The inclusion of the relaxation time and the electric displacement current modifies the governing thermal, concentration and electromagnetic field equations, changing them from the parabolic to a hyperbolic type, and thereby eliminating the unrealistic result that thermal, concentration and electromagnetic disturbances are realized instantaneously everywhere within a perfectly conducting polar fluid.

2 Mathematical formulation

In our consideration of two-dimensional problem of hydromagnetics heat and mass transfer flow, we shall make two important restrictions. First, we assume that the medium under consideration is perfectly conducting fluid and secondly, the initial magnetic field of uniform strength $\mathbf{H}_0$ acts in the direction of the y-axis. This produces an induced magnetic field $\mathbf{h}$ and induced electric field $\mathbf{E}$ which satisfy the linearized equations of electromagnetic, valid for slowly moving media of a perfect conductor [16]:

$$\operatorname{curl} h = J + \varepsilon_o \frac{\partial E}{\partial t}, \tag{2.1}$$

$$\operatorname{curl} E = -\mu_0 \frac{\partial h}{\partial t}, \tag{2.2}$$

$$E = -\mu_0(\mathbf{V} \times H_0), \tag{2.3}$$

$$\operatorname{div} h = 0, \tag{2.4}$$

where J is the electric current density, ε_o and μ_o are the magnetic and electric permeabilities, t is the time, and $\mathbf{V}$ is the velocity vector of the fluid.

The basic equations of continuity, momentum, energy and concentration for unsteady flow of polar fluids in the presence of a transversal magnetic field are

$$\operatorname{div} V = 0, \tag{2.5}$$

$$\rho \dot{V} = (\mu + \mu_r)\nabla^2 V + 2\mu_r(\nabla \times \Omega) + \mu_o(J \times H_o), \tag{2.6}$$

$$\rho k^2 \dot{\Omega} = (C_\alpha + 2C_\beta)\nabla(\nabla \cdot \Omega) - (C_\beta + C_\gamma)\nabla \times (\nabla \times \Omega), \tag{2.7}$$

$$\begin{aligned}\rho\Big(\dot{T} + \tau_o \frac{\partial \dot{T}}{\partial t}\Big) = {} & \frac{\lambda}{C_p}\nabla^2 T + \frac{(\mu + \mu_r)}{C_p}|\nabla \times V|^2 + \frac{\mu_r}{C_p}\Omega^2 + \frac{\rho D k_T}{C_s C_p}\nabla^2 C \\ & + \frac{2\mu_r}{C_p}\Omega \cdot (\nabla \times V) + \frac{\gamma^*}{C_p}|\nabla \times \Omega|^2 + k_1(T - T_\infty),\end{aligned} \tag{2.8}$$

$$\Big(\dot{C} + \tau_o \frac{\partial \dot{C}}{\partial t}\Big) = D\nabla^2 C + k_2\,(C - C_\infty) + \frac{D k_T}{T_m}\nabla^2 T, \tag{2.9}$$

where the over dot denotes the material derivative, μ is the viscosity, μ_r is the rotational viscosity, ρ is the density of the fluid, k, is the radius of gyration of the polar fluid, C_α, C_β, and C_γ are the coefficients of couple stress viscosities, Ω is the angular velocity vector of the fluid, T is the temperature of the fluid, τ_o is the relaxation time, λ is the thermal conductivity, C_p is the specific heat at constant pressure, ∇ is the del operator, D is the mass diffusivity, k_T is the thermal diffusion ratio, C_s is the concentration susceptibility, γ^* is the viscosity coefficient, k_1 is the rate of specific internal heat generation, T_∞ is the temperature condition far away from the surface, C is the concentration of the fluid, k_2 is the reaction rate coefficient, C_∞ is the concentration condition far away from the surface and T_m is the mean fluid temperature.

Let us consider the unsteady incompressible flow of two dimensional MHD polar fluid which issues from a thin slit as found on polymer processing applications past a horizontal stretching sheet. Fig. 1 shows the flow model and coordinate system. Initially we assume that the velocity of a point on a sheet is proportional to the distance

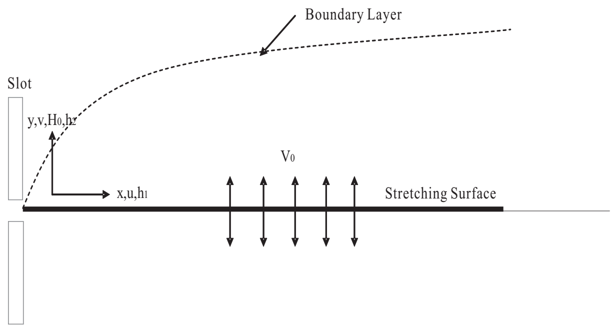

Figure 1: Coordinate system for the physical model of the stretching sheet.

from the slit. The x and y axes are taken along and perpendicular to the surface, respectively, the velocity vector $\mathbf{V}$, the angular velocity vector $\mathbf{\Omega}$, initial magnetic field $\mathbf{H}_o$, induced magnetic field $\mathbf{h}$, the induced electric field $\mathbf{E}$ which is normal to the considered magnetic field, and the electric current density $\mathbf{J}$ is parallel to the electric field as

$$\mathbf{V}=(u,v,0),\ \ \mathbf{\Omega}=(0,0,\omega),\ \ \mathbf{H}_o=(0,H_o,0),$$

$$\mathbf{h}=(h_1,h_2,0),\ \ \mathbf{E}=(0,0,E)\ \text{ and }\ \mathbf{J}=(0,0,J).$$

With the usual boundary layer assumptions [20,21], Eqs. (2.5)-(2.9) are reduced to the following:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.10}$$

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=&(\nu+\nu_r)\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big)+2\nu_r\frac{\partial \omega}{\partial y}\\&+\frac{\alpha^2}{H_o}\Big(\frac{\partial h_1}{\partial y}-\frac{\partial h_2}{\partial x}-\mu_o\varepsilon_o H_o\frac{\partial u}{\partial t}\Big),\end{aligned} \tag{2.11}$$

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\frac{\gamma}{I}\Big(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\Big), \tag{2.12}$$

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=&\frac{\lambda}{\rho C_p}\Big(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\Big)+\frac{(\nu+\nu_r)}{C_p}\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)^2\\&+2\nu_r\omega\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)+\frac{\gamma^*}{\rho C_p}\Big[\Big(\frac{\partial \omega}{\partial x}\Big)^2+\Big(\frac{\partial \omega}{\partial y}\Big)^2\Big]+\nu_r\omega^2+k_1(T-T_\infty)\\&-\tau_0\frac{\partial}{\partial t}\Big(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\Big)+\frac{Dk_T}{C_sC_p}\Big(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\Big),\end{aligned} \tag{2.13}$$

$$\begin{aligned}\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=&D\Big(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\Big)-\tau_o\frac{\partial}{\partial t}\Big(\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}\Big)\\&+k_2(C-C_\infty)+\frac{Dk_T}{T_m}\Big(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\Big),\end{aligned} \tag{2.14}$$

$$\frac{\partial h_1}{\partial t} = H_o \frac{\partial u}{\partial y}, \tag{2.15}$$

$$\frac{\partial h_2}{\partial t} = -H_o \frac{\partial u}{\partial x}, \tag{2.16}$$

where ν is the kinematics viscosity, ν_r is the rotational kinematics viscosity, $\alpha^2 = \mu_o H_o^2/\rho$ is the Alfven velocity, $\gamma = (C_\beta + C_\gamma)$ is the spin-gradient density and I is the scalar constant of dimension, equal to the moment of inertia of unit.

Eliminating h_1 and h_2 among Eqs. (2.15), (2.16) and (2.11), and taking into account the boundary layer approximations, Eq. (2.11) yields [21]:

$$\begin{aligned}&(1+\alpha^2\mu_o\varepsilon_o)\frac{\partial^2 u}{\partial t^2} + u\frac{\partial^2 u}{\partial t\partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial t\partial y} + \frac{\partial v}{\partial t}\frac{\partial u}{\partial y}\\ =&2\nu_r\frac{\partial^2 \omega}{\partial t\partial y} + \left((\nu+\nu_r)\frac{\partial}{\partial t} + \alpha^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).\end{aligned} \tag{2.17}$$

The initial and boundary conditions imposed on Eqs. (2.12)-(2.17), are [3]:

$$y=0:\quad u=\ell x,\quad \omega' = -u'',\quad v = V_o,\quad t=0, \tag{2.18a}$$
$$y=0:\quad C - C_\infty = C_\omega x^m,\quad T - T_\infty = T_\omega x^m,\quad t=0, \tag{2.18b}$$
$$y=0:\quad u=\ell x e^{nt},\quad \omega' = -u'' e^{nt},\quad v = V_o e^{nt},\quad t>0, \tag{2.18c}$$
$$y=0:\quad C - C_\infty = C_\omega x^m e^{nt},\quad T - T_\infty = T_\omega x^m e^{nt},\quad t>0, \tag{2.18d}$$
$$y\to\infty:\ u\to 0,\quad C\to C_\infty,\quad T\to T_\infty,\quad \omega\to 0, \tag{2.18e}$$

where ℓ is the proportional coefficient, n is constant, V_o is the velocity condition at the surface, T_ω is the mean temperature of the surface, C_ω is the mean consternation of the surface, m is the power law exponent, and primes indicate derivates with respect to y.

We introduce the following non-dimensional quantities:

$$x^* = \sqrt{\frac{\ell}{\nu}}x,\quad y^* = \sqrt{\frac{\ell}{\nu}}y,\quad t^* = \ell t,\quad h_1^* = \frac{h_1}{H_o},\quad h_2^* = \frac{h_2}{H_o},\quad u^* = \frac{u}{\sqrt{\ell\nu}}, \tag{2.19a}$$
$$v^* = \frac{v}{\sqrt{\ell\nu}},\quad P_r = \frac{\rho C_p \nu}{\lambda},\quad T^* = \frac{T-T_\infty}{T_\omega},\quad C^* = \frac{C-C_\infty}{C_\omega}, \tag{2.19b}$$
$$\alpha^* = \frac{\alpha}{\sqrt{\ell\nu}},\quad \beta = \frac{I\nu}{\gamma},\quad \alpha_1 = \frac{\nu_r}{\nu},\quad E_c = \frac{\ell\nu(1+\alpha_1)}{T_\omega C_p},\quad f_\omega = \frac{V_o}{\sqrt{\ell\nu}}, \tag{2.19c}$$
$$\omega^* = \frac{\omega}{\ell},\quad E_1 = \frac{\ell\nu\alpha_1}{T_\omega C_p},\quad E_2 = \frac{\gamma\ell^2}{\rho T_\omega C_p},\quad D_f = \frac{Dk_T C_\omega}{C_s C_p \nu T_\omega},\quad k_1^* = \frac{k_1}{\ell}, \tag{2.19d}$$
$$k_2^* = \frac{k_2}{\ell},\quad S_c = \frac{\nu}{D},\quad S_r = \frac{Dk_T T_\omega}{\nu C_\omega T_m},\quad \tau_o^* = \ell\tau_o, \tag{2.19e}$$

where P_r is the Prandtl number, β, α_1 are the material parameters characterizing the polarity of the fluid, E_c, E_1, and E_2 are the Eckert numbers, D_f is the Dufour number, S_c is the Schmidt number, k_1^* is the thermal radiation parameter, k_2^* is the first order of reaction (chemically reactive), and S_r is the Soret number.

The mass transfer parameter f_ω is positive for injection and negative for suction. Invoking the non-dimensional quantities above, Eqs. (2.10), (2.12)-(2.14) and (2.17), are reduced to the non-dimensional equations, dropping the asterisks for convenience,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.20}$$

$$\begin{aligned} a_1\frac{\partial^2 u}{\partial t^2}+u\frac{\partial^2 u}{\partial t\partial x}+\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial v}{\partial t}\frac{\partial u}{\partial y} \\ =2\alpha_1\frac{\partial^2\omega}{\partial t\partial y}+\Big((1+\alpha_1)\frac{\partial}{\partial t}+\alpha^2\Big)\Big(\frac{\partial^2 u}{\partial y^2}\Big), \end{aligned} \tag{2.21}$$

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{\beta}\frac{\partial^2\omega}{\partial y^2}, \tag{2.22}$$

$$\begin{aligned} \frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{P_r}\frac{\partial^2 T}{\partial y^2}+D_f\frac{\partial^2 C}{\partial y^2}+E_c\Big(\frac{\partial u}{\partial y}\Big)^2+E_1\omega^2-2E_1\omega\frac{\partial u}{\partial y} \\ +k_1T+E_2\Big(\frac{\partial\omega}{\partial y}\Big)^2-\tau_0\frac{\partial}{\partial t}\Big(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\Big), \end{aligned} \tag{2.23}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{1}{Sc}\frac{\partial^2 C}{\partial y^2}+k_2C+S_r\frac{\partial^2 T}{\partial y^2}-\tau_o\frac{\partial}{\partial t}\Big(\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}\Big), \tag{2.24}$$

where $a_1=\alpha^2/c^2$ and c is the speed of light given by $c^2=1/\mu\varepsilon_o$.

From Eq. (2.23) the reduced boundary conditions are

$$\begin{array}{lllll} y=0: & u=x, & \omega'=-u'', & v=f_\omega, & t=0, & (2.25a)\\ y=0: & C=x^m, & T=x^m, & & t=0, & (2.25b)\\ y=0: & u=xe^{nt}, & \omega'=-u'', & v=f_\omega e^{nt}, & t>0, & (2.25c)\\ y=0: & C=x^me^{nt}, & T=x^me^{nt}, & & t>0, & (2.25d)\\ y\to\infty & u\to 0, & C\to 0, & \omega\to 0, & T\to T_\infty. & (2.25e) \end{array}$$

3 The method of successive approximations

A process of successive approximations will integrate the unsteady boundary layer (2.20)-(2.24). Selecting a system of coordinates at rest with respect to the plate and the MHD flow of perfectly conducting fluid moves with respect to the plane surface, we can assume that the velocity u, v, the temperature T, and the concentration C, which possess a series solution of the form

$$F(x,y,t)=\sum_{i=0}^{\infty}F_i(x,y,t), \tag{3.1}$$

where $F_i=O(\varepsilon^i)$ i – integer and ε is a small number.

Substituting the series (3.1) into Eqs.(2.21)-(2.24) and setting equal to zero terms of the same order, one obtains equations to find components of the series (3.1)

$$\left\{(1+\alpha_1)\frac{\partial}{\partial t}+\alpha^2\right\}\frac{\partial^2 u_o}{\partial y^2}-a_1\frac{\partial^2 u_o}{\partial t^2}=-2\alpha_1\frac{\partial^2 \omega_o}{\partial t\partial y}, \tag{3.2}$$

$$\left\{(1+\alpha_1)\frac{\partial}{\partial t}+\alpha^2\right\}\frac{\partial^2 u_1}{\partial y^2}-a_1\frac{\partial^2 u_1}{\partial t^2}=\frac{\partial}{\partial t}\left(u_o\frac{\partial u_o}{\partial x}+v_o\frac{\partial u_o}{\partial y}\right)-2\alpha_1\frac{\partial^2 \omega_o}{\partial t\partial y}, \tag{3.3}$$

$$\frac{\partial^2 \omega_o}{\partial y^2}-\beta\frac{\partial \omega_o}{\partial t}=0, \tag{3.4}$$

$$\frac{\partial^2 \omega_1}{\partial y^2}-\beta\frac{\partial \omega_1}{\partial t}=\beta\left(u_o\frac{\partial \omega_o}{\partial x}+v_o\frac{\partial \omega_o}{\partial y}\right), \tag{3.5}$$

$$\frac{\partial^2 T_o}{\partial y^2}-P_r\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial T_o}{\partial t}+P_rk_1T_o+P_rD_f\frac{\partial^2 C_o}{\partial y^2}=0, \tag{3.6}$$

$$\begin{aligned}&\frac{\partial^2 T_1}{\partial y^2}+P_rD_f\frac{\partial^2 C_1}{\partial y^2}-P_r\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial T_1}{\partial t}+P_rk_1T_1\\=&-E_1\omega_o^2-E_c\left(\frac{\partial u_0}{\partial y}\right)^2+P_r\left(1+\tau_o\frac{\partial}{\partial t}\right)\left\{u_o\frac{\partial T_o}{\partial x}+v_o\frac{\partial T_o}{\partial y}\right\}\\&-2E_1\omega_o\frac{\partial u_0}{\partial y}-E_2\frac{\partial \omega_o}{\partial y},\end{aligned} \tag{3.7}$$

$$\frac{\partial^2 C_o}{\partial y^2}+S_cS_r\frac{\partial^2 T_0}{\partial y^2}-S_c\left(1+\tau_o\frac{\partial}{\partial t}\right)\frac{\partial C_o}{\partial t}+S_ck_2C_o=0, \tag{3.8}$$

$$\begin{aligned}&\frac{\partial^2 C_1}{\partial y^2}+S_cS_r\frac{\partial^2 T_1}{\partial y^2}-S_c\left(1+\tau_o\frac{\partial}{\partial t}\right)\frac{\partial C_1}{\partial t}+S_ck_2C_1\\=&S_c\left(1+\tau_o\frac{\partial}{\partial t}\right)\left\{u_o\frac{\partial C_o}{\partial x}+v_o\frac{\partial C_o}{\partial y}\right\}.\end{aligned} \tag{3.9}$$

Comparing Eq. (2.25) to Eq. (3.1) , we will have the corresponding boundary conditions

$$\begin{array}{llllll}
y=0: & u_o=xe^{nt}, & u_i=0, & i=1,2,\cdots, & t>0, & \text{(3.10a)}\\
y=0: & v_o=f_\omega e^{nt}, & v_i=0, & i=1,2,\cdots, & t>0, & \text{(3.10b)}\\
y=0: & \omega_i=-u_i'', & & i=0,1,2,\cdots, & t>0, & \text{(3.10c)}\\
y=0: & T_o=x^me^{nt}, & T_i=0, & i=1,2,\cdots, & t>0, & \text{(3.10d)}\\
y=0: & C_o=x^me^{nt}, & C_i=0, & i=1,2,\cdots, & t>0, & \text{(3.10e)}\\
y\to\infty: & u_i\to 0, & \omega_i\to 0, & i=0,1,2,\cdots, & & \\
y\to\infty: & T_i\to 0, & C_i\to 0, & i=0,1,2,\cdots. & &
\end{array}$$

In the following analysis, the first two terms in the solution series (3.1) will be retained. It is known as a fact that such solution is satisfactory in the phases of the non-periodic motion after it has been stated from the rest (till the moment when the first separation of boundary layer occurs) and in the case of periodic motion when the amplitude of oscillation is small. Higher-order approximations u_3, can be obtained easily in principle. However, the complexity of the successive approximations method increases rapidly as higher approximations are considered. It is also known that the third and higher terms series solutions give small changes in the results compared with the two terms series solutions.

4 Solution of the momentum and angular momentum equations

Let us suppose that the exact solutions of the differential Eqs. (3.3) and (3.4) are of the form

$$u_o(x,y,t) = xe^{nt} f_o'(y), \tag{4.1}$$
$$\omega_o(x,y,t) = xe^{nt} \varphi_o(y). \tag{4.2}$$

Using Eq. (2.20), we obtain,

$$v_o(x,y,t) = -e^{nt} f_o(y), \tag{4.3}$$

from Eqs. (3.2) and (3.4) and by using Eqs. (4.1)-(4.3), one obtains the differential equations of the unknown functions $f_o(y)$, $\varphi_o(y)$, and the corresponding boundary conditions

$$f_o''' - r_1^2 f_o' = -\frac{2\alpha_1}{na_1} r_1^2 \varphi_o', \tag{4.4}$$
$$\varphi_o'' - n\beta\varphi_o = 0, \tag{4.5}$$
$$\begin{aligned} y = 0: &\qquad f_o = -f_\omega, \qquad f_o' = 1, \qquad \varphi_0' = -f_0''', \\ y \to \infty: &\qquad f_0' \to 0, \qquad \varphi_o \to 0, \end{aligned} \tag{4.6}$$

where $r_1^2 = n^2 a_1/(1+\alpha_1)n + \alpha^2$.

The solutions of the system (4.4)-(4.6) are of the form

$$f_o(y) = A_1 + A_2 e^{-r_1 y} + AA_3 e^{-\sqrt{n\beta} y}, \tag{4.7}$$
$$\varphi_o(y) = A_3 e^{-\sqrt{n\beta} y}, \tag{4.8}$$

where

$$A = \frac{2\alpha_1 r_1^2}{na_1\left(k_1^2 - n\beta\right)}, \qquad A_1 = -\left(f_\omega + A_2 + AA_3\right),$$
$$A_2 = \frac{nA\beta + 1}{k_1\left[A\left(r_1^2 - n\beta\right) - 1\right]}, \quad A_3 = \frac{k_1^2}{\sqrt{n\beta}\left[1 + A\left(n\beta - r_1^2\right)\right]}.$$

Assuming the solutions of the differential Eqs. (3.3) and (3.5) are of the form

$$u_1(x,y,t) = xe^{2nt} f_1'(y), \tag{4.9}$$

$$\omega_1(x,y,t) = xe^{2nt} \varphi_1(y), \tag{4.10}$$

and using Eqs. (4.1)-(4.3), (4.9) and (4.10), one obtains from the Eqs. (3.3) and (3.5), the differential equations for $f_1(y)$, $\varphi_1(y)$, and the corresponding boundary conditions

$$f_1''' - r_2^2 f_1' = \frac{r_2^2}{2na_1}\left(f_o'^2 - f_o f_o'' - 2\alpha_1 \varphi_o'\right), \tag{4.11}$$

$$\varphi_1'' - n\beta\varphi_1 = \beta\left(\varphi_o f_o' - f_o \varphi_o'\right), \tag{4.12}$$

$$\begin{aligned} &y = 0: \quad && f_1 = 0, \quad && f_1' = 0, \quad && \varphi_1' = -f_1''', \\ &y \to \infty: \quad && f_1' \to 0, \quad && \varphi_1 \to 0, \end{aligned} \tag{4.13}$$

where $r_2^2 = 4n^2 a_1 / 2n(1+\alpha_1) + \alpha^2$.

Using Eqs. (4.7) and (4.8), one obtains the solutions of the system (4.11)-(4.13)

$$f_1(y) = B_o + B_1 e^{-r_1 y} + B_2 e^{-\sqrt{n\beta}y} + B_3 e^{-r_2 y} + B_4 e^{-(r_1+\sqrt{n\beta})y} + BB_5 e^{-2\sqrt{n\beta}y}, \tag{4.14}$$

$$\varphi_1(y) = B_5 e^{-2\sqrt{n\beta}y} + B_6 e^{-\sqrt{n\beta}y} + B_7 e^{-(r_1+\sqrt{n\beta})y}, \tag{4.15}$$

where

$$B = \frac{2\alpha_1 r_2^2}{na_1\left(r_2^2 - 2n\beta\right)}, \qquad B_1 = \frac{r_1 r_2^2}{2na_1\left(r_1^2 - r_2^2\right)} A_1 A_2,$$

$$B_2 = \frac{r_2^2\left(n\beta A A_1 A_3 - 2\alpha_1\sqrt{n\beta}B_6\right)}{2na_1\sqrt{n\beta}\left(n\beta - r_2^2\right)}.$$

We can obtain an exact solution of Eqs. (3.6) and (3.8) if we consider the case $m = 2$. Suppose that the exact solutions of these differential equations are of the form

$$T_o(x,y,t) = x^2 e^{nt}\psi_o(y), \qquad C_o(x,y,t) = x^2 e^{nt}\zeta_o(y), \tag{4.16}$$

from Eqs. (3.6) and (3.8), by using Eq. (4.16) one obtains the differential equations of the unknown functions $\psi_o(y)$ and $\zeta_o(y)$, and the corresponding boundary conditions:

$$\psi_o'' - P_r\left[n\left(1+n\tau_o\right) - k_1\right]\psi_o = -P_r D_f \zeta_o'', \tag{4.17}$$

$$\zeta_o'' - S_c\left[n\left(1+n\tau_o\right) - k_2\right]\zeta_o = -S_c S_r \psi_o'', \tag{4.18}$$

$$\begin{aligned} &y = 0: \quad && \psi_o = 1, \quad && \zeta_o = 1, \\ &y \to \infty: \quad && \psi_o \to 0, \quad && \zeta_o \to 0. \end{aligned} \tag{4.19}$$

It is obvious that the two unknown functions $\psi_o(y)$ and $\zeta_o(y)$ satisfy with the fourth order differential equations

$$\psi_o^{iv} - \left(r_3^2 + r_4^2\right)\psi_o'' - r_3^2 r_4^2 \psi_o = 0, \tag{4.20}$$

$$\zeta_o^{iv} - \left(r_3^2 + r_4^2\right)\zeta_o'' - r_3^2 r_4^2 \zeta_o = 0, \tag{4.21}$$

where

$$r_3^2 + r_4^2 = \frac{n(1+n\tau_o)(P_r+S_c) - P_r k_1 - S_c k_2}{1 - P_r D_f S_c S_r},$$

$$r_3^2 r_4^2 = \frac{P_r S_c \big[n(1+n\tau_o) - k_1\big]\big[n(1+n\tau_o) - k_2\big]}{1 - P_r D_f S_c S_r}.$$

The solutions of the system (4.20) and (4.21) are of the form

$$\psi_o(y) = \Gamma_1 e^{-r_3 y} + \Gamma_2 e^{-r_4 y}, \tag{4.22}$$

$$\zeta_o(y) = \ell_1 \Gamma_1 e^{-r_3 y} + \ell_2 \Gamma_2 e^{-r_4 y}, \tag{4.23}$$

where Γ_1, Γ_2, ℓ_1, and ℓ_2 are some unknown parameters. We can determent the parameters ℓ_1 and ℓ_2 from the compatibility between Eqs. (4.17) and (4.18) gives

$$\ell_1 = \frac{P - r_3^2}{P_r D_f}, \qquad \ell_2 = \frac{P - r_4^2}{P_r D_f}, \qquad P = P_r\left[n\left(1 + n\tau_o\right) - k_1\right].$$

We shall now use the boundary conditions of the problem to evaluate the unknown parameters Γ_1 and Γ_2. Eq. (4.14) together with Eqs. (4.15) and (4.16) which give

$$\Gamma_1 = \frac{P - r_4^2 - P_r D_f}{r_3^2 - r_4^2}, \qquad \Gamma_2 = \frac{P - r_3^2 - P_r D_f}{r_4^2 - r_3^2}.$$

Now assuming the solutions of the differential Eqs. (3.7) and (3.9) are of the form

$$T_1(x,y,t) = x^2 e^{2nt}\psi_1(y), \qquad C_1(x,y,t) = x^2 e^{2nt}\zeta_1(y). \tag{4.24}$$

and using Eqs. (4.21) and (4.27), one obtains from Eqs. (4.2) and (4.4) the differential equations for $\psi_1(y)$, $\zeta_1(y)$, and the corresponding boundary conditions:

$$\begin{aligned}
\psi_1'' - P_1\psi_1 + P_r D_f \zeta_1'' &= -E_1\left(\varphi_o^2 + 2\varphi_o f_o''\right) - E_2\left(f_o''^2 + \varphi_o'^2\right) \\
&\quad + P_r\left(1 + 2n\tau_o\right)\left(2f_o'\psi_o - f_o\psi_o'\right), && (4.25)\\
\zeta_1'' - S_1\zeta_1 + S_c S_r \psi_1'' &= S_c\left(1 + 2n\tau_o\right)\left(2f_o'\zeta_o - f_o\zeta_o'\right), && (4.26)\\
y = 0: &\qquad \psi_1 = 1, \qquad \zeta_1 = 0, \\
y \to \infty: &\qquad \psi_1 \to 0, \qquad \zeta_1 \to 0, && (4.27)
\end{aligned}$$

where $P_1 = P_r\big(2n(1+2n\tau_o) - k_1\big)$, $S_1 = S_c\big(2n(1+2n\tau_o) - k_2\big)$.

Eliminating ζ_1 from Eqs. (4.25) and (4.26), after substituting it from Eqs. (4.7), (4.8), (4.20) and (4.21), we obtain

$$\begin{aligned}
&\psi_1^{iv} - \left(r_5^2 + r_6^2\right)\psi_1'' - r_5^2 r_6^2 \psi_1 \\
=&m_1 e^{-2\sqrt{n\beta}y} + m_2 e^{-\left(r_1 + \sqrt{n\beta}\right)y} + m_3 e^{-\left(r_3 + \sqrt{n\beta}\right)y} + m_4 e^{-\left(r_4 + \sqrt{n\beta}\right)y} \\
&\quad + m_5 e^{-(r_1+r_3)y} + m_6 e^{-(r_1+r_4)y} + m_7 e^{-2r_1 y} + m_8 e^{-r_3 y} + m_9 e^{-r_4 y}, \qquad (4.28)
\end{aligned}$$

where

$$
\begin{aligned}
r_5^2+r_6^2 &= \frac{P_1+S_1}{1-P_rD_fS_cS_r}, \qquad r_5^2r_6^2 = \frac{P_1S_1}{1-P_rD_fS_cS_r},\\
m_1 &= \frac{A_3^2\left(S_1-4n\beta\right)\left[E_1\left(1+2An\beta\right)+n\beta E_2\left(1+A^2n\beta\right)\right]}{\left(1-P_rD_fS_cS_r\right)},\\
m_2 &= \frac{2r_1^2A_2A_3\left(S_1-\left(r_1+\sqrt{n\beta}\right)^2\right)\left(E_1+An\beta E_2\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_3 &= \frac{AA_3\Gamma_1P_r\left(1+2n\tau_0\right)\left(\left(S_c\ell_1D_f-1\right)\left(r_3+\sqrt{n\beta}\right)^2+S_1\right)\left(r_3-2\sqrt{n\beta}\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_4 &= \frac{AA_3\Gamma_2P_r\left(1+2n\tau_0\right)\left(\left(S_c\ell_1D_f-1\right)\left(r_4+\sqrt{n\beta}\right)^2+S_1\right)\left(r_4-2\sqrt{n\beta}\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_5 &= \frac{A_2\Gamma_1P_r\left(1+2n\tau_0\right)\left(\left(1-S_c\ell_1D_f\right)\left(r_1+r_3\right)^2-S_1\right)\left(r_3-2r_1\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_6 &= \frac{A_2\Gamma_2P_r\left(1+2n\tau_0\right)\left(\left(S_c\ell_2D_f-1\right)\left(r_1+r_4\right)^2+S_1\right)\left(r_4-2r_1\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_7 &= \frac{E_2A_2^2r_1^4\left(S_1-4r_1^2\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_8 &= \frac{A_1r_3\Gamma_1P_r\left(1+2n\tau_0\right)\left(r_3^2\left(1-S_c\ell_1D_f\right)-S_1\right)}{\left(1-P_rD_fS_cS_r\right)},\\
m_9 &= \frac{A_1r_4\Gamma_2P_r\left(1+2n\tau_0\right)\left(r_4^2\left(1-S_c\ell_2D_f\right)-S_1\right)}{\left(1-P_rD_fS_cS_r\right)}.
\end{aligned}
$$

Eliminating ψ_1 between Eqs. (4.25) and (4.26) after substituting it from Eqs. (4.7), (4.8), (4.20) and (4.21), we obtain

$$
\begin{aligned}
&\zeta_1^{iv}-\left(r_5^2+r_6^2\right)\zeta_1''-r_5^2r_6^2\zeta_1\\
=&m_{10}e^{-2\sqrt{n\beta}y}+m_{11}e^{-\left(r_1+\sqrt{n\beta}\right)y}+m_{12}e^{-\left(r_3+\sqrt{n\beta}\right)y}\\
&+m_{13}e^{-\left(r_4+\sqrt{n\beta}\right)y}+m_{14}e^{-(r_1+r_3)y}+m_{15}e^{-(r_1+r_4)y}\\
&+m_{16}e^{-2r_1y}+m_{17}e^{-r_3y}+m_{18}e^{-r_4y},
\end{aligned}
\tag{4.29}
$$

where

$$
\begin{aligned}
m_{10} &= \frac{4n\beta A_3^2S_cS_r\left[E_1\left(1+2An\beta\right)+n\beta E_2\left(1+A^2n\beta\right)\right]}{\left(1-P_rD_fS_cS_r\right)},\\
m_{11} &= \frac{2r_1^2S_cS_rA_2A_3\left(r_1+\sqrt{n\beta}\right)^2\left(E_1+An\beta E_2\right)}{\left(1-P_rD_fS_cS_r\right)},
\end{aligned}
$$

$$m_{12} = \frac{AA_3\Gamma_1 S_c\,(1+2n\tau_0)\left(\left(\ell_1 - S_r P_r\right)\left(r_3+\sqrt{n\beta}\right)^2 + \ell_1 P_1\right)\left(r_3 - 2\sqrt{n\beta}\right)}{\left(1 - P_r D_f S_c S_r\right)},$$

$$m_{13} = \frac{AA_3\Gamma_2 S_c\,(1+2n\tau_0)\left(\left(\ell_2 - S_r P_r\right)\left(r_4+\sqrt{n\beta}\right)^2 + \ell_2 P_1\right)\left(r_4 - 2\sqrt{n\beta}\right)}{\left(1 - P_r D_f S_c S_r\right)},$$

$$m_{14} = \frac{A_2\Gamma_1 S_c\,(1+2n\tau_0)\left(\left(\ell_1 - S_r P_r\right)\left(r_1+r_3\right)^2 + \ell_1 P_1\right)\left(r_3 - 2r_1\right)}{\left(1 - P_r D_f S_c S_r\right)},$$

$$m_{15} = \frac{A_2\Gamma_2 S_c\,(1+2n\tau_0)\left(\left(\ell_2 - S_r P_r\right)\left(r_1+r_4\right)^2 + \ell_2 P_1\right)\left(r_4 - 2r_1\right)}{\left(1 - P_r D_f S_c S_r\right)},$$

$$m_{16} = \frac{4E_2 A_2^2 r_1^6 S_c S_r}{\left(1 - P_r D_f S_c S_r\right)},$$

$$m_{17} = \frac{A_1 r_3 \Gamma_1 S_c\,(1+2n\tau_0)\left(r_3^2\left(\ell_1 - S_r P_r\right) - S_1\right)}{\left(1 - P_r D_f S_c S_r\right)},$$

$$m_{18} = \frac{A_1 r_4 \Gamma_2 S_c\,(1+2n\tau_0)\left(r_4^2\left(\ell_2 - S_r P_r\right) - P_1\right)}{\left(1 - P_r D_f S_c S_r\right)}.$$

The solutions of the Eqs. (4.31) and (4.32) are of the form

$$\begin{aligned}
\psi_1(y) =& \Gamma_3 e^{-r_5 y} + \Gamma_4 e^{-r_6 y} + \widetilde{m}_1 e^{-2\sqrt{n\beta}y} + \widetilde{m}_2 e^{-\left(r_1+\sqrt{n\beta}\right)y} + \widetilde{m}_3 e^{-\left(r_3+\sqrt{n\beta}\right)y} \\
&+ \widetilde{m}_4 e^{-\left(r_4+\sqrt{n\beta}\right)y} + \widetilde{m}_5 e^{-(r_1+r_3)y} + \widetilde{m}_6 e^{-(r_1+r_4)y} \\
&+ \widetilde{m}_7 e^{-2r_1 y} + \widetilde{m}_8 e^{-r_3 y} + \widetilde{m}_9 e^{-r_4 y},
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\zeta_1(y) =& \Gamma_3 \ell_3 e^{-r_5 y} + \Gamma_4 \ell_4 e^{-r_6 y} + \widetilde{m}_{10} e^{-2\sqrt{n\beta}y} + \widetilde{m}_{11} e^{-\left(r_1+\sqrt{n\beta}\right)y} \\
&+ \widetilde{m}_{12} e^{-\left(r_3+\sqrt{n\beta}\right)y} + \widetilde{m}_{13} e^{-\left(r_4+\sqrt{n\beta}\right)y} + \widetilde{m}_{14} e^{-(r_1+r_3)y} \\
&+ \widetilde{m}_{15} e^{-(r_1+r_4)y} + \widetilde{m}_{16} e^{-2r_1 y} + \widetilde{m}_{17} e^{-r_3 y} + \widetilde{m}_{18} e^{-r_4 y},
\end{aligned} \tag{4.31}$$

where

$$\widetilde{m}_1 = \frac{m_1}{\left(4n\beta - r_5^2\right)\left(4n\beta - r_6^2\right)},$$

$$\widetilde{m}_2 = \frac{m_2}{\left(r_1^2 + 2r_1\sqrt{n\beta} + n\beta - r_5^2\right)\left(r_1^2 + 2r_1\sqrt{n\beta} + n\beta - r_6^2\right)},$$

$$\widetilde{m}_3 = \frac{m_3}{\left(r_3^2 + 2r_3\sqrt{n\beta} + n\beta - r_5^2\right)\left(r_3^2 + 2r_3\sqrt{n\beta} + n\beta - r_6^2\right)},$$

$$\widetilde{m}_4 = \frac{m_4}{\left(r_4^2 + 2r_4\sqrt{n\beta} + n\beta - r_5^2\right)\left(r_4^2 + 2r_4\sqrt{n\beta} + n\beta - r_6^2\right)},$$

$$\widetilde{m}_5 = \frac{m_5}{\left(r_1^2 + 2r_1 r_3 + r_3^2 - r_5^2\right)\left(r_1^2 + 2r_1 r_3 + r_3^2 - r_6^2\right)},$$

$$
\begin{aligned}
&\widetilde{m}_6=\frac{m_6}{\left(r_1^2+2r_1r_4+r_4^2-r_5^2\right)\left(r_1^2+2r_1r_4+r_4^2-r_6^2\right)},\\
&\widetilde{m}_7=\frac{m_7}{\left(4r_1^2-r_5^2\right)\left(4r_1^2-r_6^2\right)}, \qquad \widetilde{m}_8=\frac{m_8}{\left(r_3^2-r_5^2\right)\left(r_3^2-r_6^2\right)},\\
&\widetilde{m}_9=\frac{m_9}{\left(r_4^2-r_5^2\right)\left(r_4^2-r_6^2\right)}, \qquad \widetilde{m}_{10}=\frac{m_{10}}{\left(4n\beta-r_5^2\right)\left(4n\beta-r_6^2\right)},\\
&\widetilde{m}_{11}=\frac{m_{11}}{\left(r_1^2+2r_1\sqrt{n\beta}+n\beta-r_5^2\right)\left(r_1^2+2r_1\sqrt{n\beta}+n\beta-r_6^2\right)},\\
&\widetilde{m}_{12}=\frac{m_{12}}{\left(r_3^2+2r_3\sqrt{n\beta}+n\beta-r_5^2\right)\left(r_3^2+2r_3\sqrt{n\beta}+n\beta-r_6^2\right)},\\
&\widetilde{m}_{13}=\frac{m_{13}}{\left(r_4^2+2r_4\sqrt{n\beta}+n\beta-r_5^2\right)\left(r_4^2+2r_4\sqrt{n\beta}+n\beta-r_6^2\right)},\\
&\widetilde{m}_{14}=\frac{m_{14}}{\left(r_1^2+2r_1r_3+r_3^2-r_5^2\right)\left(r_1^2+2r_1r_3+r_3^2-r_6^2\right)},\\
&\widetilde{m}_{15}=\frac{m_{15}}{\left(r_1^2+2r_1r_4+r_4^2-r_5^2\right)\left(r_1^2+2r_1r_4+r_4^2-r_6^2\right)},\\
&\widetilde{m}_{16}=\frac{m_{16}}{\left(4r_1^2-r_5^2\right)\left(4r_1^2-r_6^2\right)}, \quad \widetilde{m}_{17}=\frac{m_{17}}{\left(r_3^2-r_5^2\right)\left(r_3^2-r_6^2\right)},\\
&\widetilde{m}_{18}=\frac{m_{18}}{\left(r_4^2-r_5^2\right)\left(r_4^2-r_6^2\right)}.
\end{aligned}
$$

We can determent the parameters ℓ_3, and ℓ_4 from the compatibility between Eqs. (4.25) and (4.26), which give

$$
\ell_3=\frac{P_1-r_5^2}{P_rD_f}, \qquad \ell_4=\frac{P_1-r_6^2}{P_rD_f}.
$$

We shall now use the boundary conditions of the problem to evaluate the unknown parameters Γ_3 and Γ_4. Eqs. (4.27) together with Eqs. (4.30) and (4.31) gives

$$
\Gamma_3=\frac{m_jP_rD_f-m_i(P_1-r_6^2)}{r_5^2-r_6^2}, \qquad \Gamma_4=\frac{m_jP_rD_f-m_i(P_1-r_5^2)}{r_6^2-r_5^2},
$$

where $i=1,2,\cdots,9,\ j=10,11,\cdots,18$.

From Eqs. (2.15) and (2.16) by virtue of transform Eq. (2.19), we get

$$
\frac{\partial h_1}{\partial t}=\frac{\partial u}{\partial y}, \qquad \frac{\partial h_2}{\partial t}=-\frac{\partial u}{\partial x}. \tag{4.32}
$$

By using Eqs. (4.1), (4.7), (4.9), (4.14) and (4.32), the components of the induced magnetic field are given by

$$
\begin{aligned}
h_1(x,y,t)&=h_{01}(x,y,t)+\varepsilon h_{11}(x,y,t), &(4.33)\\
h_2(x,y,t)&=h_{02}(x,y,t)+\varepsilon h_{12}(x,y,t), &(4.34)
\end{aligned}
$$

where

$$
\begin{aligned}
h_{01}(x,y,t) &= \frac{xe^{nt}}{n}\left(r_1^2 A_2 e^{-r_1 y} + n\beta A A_3 e^{-\sqrt{n\beta}y}\right),\\
h_{11}(x,y,t) &= \frac{xe^{2nt}}{2n}\left(r_1^2 B_1 e^{-r_1 y} + n\beta B_2 e^{-\sqrt{n\beta}y} + r_2^2 B_3 e^{-r_2 y}\right.\\
&\qquad \left. + (r_1 + n\beta)^2 B_4 e^{-\left(r_1+\sqrt{n\beta}\right)y} + 4n\beta B B_5 e^{-\sqrt{2n\beta}y}\right),\\
h_{02}(x,y,t) &= \frac{e^{nt}}{n}\left(r_1 A_2 e^{-r_1 y} + n\beta A A_3 e^{-\sqrt{n\beta}y}\right),\\
h_{12}(x,y,t) &= \frac{e^{2nt}}{2n}\left(r_1 B_1 e^{-r_1 y} + \sqrt{n\beta} B_2 e^{-\sqrt{n\beta}y} + r_2 B_3 e^{-r_2 y}\right.\\
&\qquad \left. + (r_1 + n\beta) B_4 e^{-\left(r_1+\sqrt{n\beta}\right)y} + 2\sqrt{n\beta} B B_5 e^{-2\sqrt{n\beta}y}\right).
\end{aligned}
$$

From Eqs. (2.1), (2.3), (2.25), (4.1), (4.7), (4.9) and (4.14), by virtue of transform Eq. (2.19), the electric field and electric current density are given by

$$E(x,y,t) = E_0(x,y,t) + \varepsilon E_1(x,y,t), \tag{4.35}$$

$$J(x,y,t) = J_0(x,y,t) + \varepsilon J_1(x,y,t), \tag{4.36}$$

where

$$
\begin{aligned}
E_0(x,y,t) &= xe^{nt}\left(r_1 A_2 e^{-r_1 y} + \sqrt{n\beta} A A_3 e^{-\sqrt{n\beta}y}\right),\\
E_1(x,y,t) &= xe^{2nt}\left(r_1 B_1 e^{-r_1 y} + \sqrt{n\beta} B_2 e^{-\sqrt{n\beta}y} + r_2 B_3 e^{-r_2 y}\right.\\
&\qquad \left. + \left(r_1 + \sqrt{n\beta}\right) B_4 e^{-\left(r_1+\sqrt{n\beta}\right)y} + 2\sqrt{n\beta} B B_5 e^{-2\sqrt{n\beta}y}\right),\\
J_0(x,y,t) &= \frac{e^{nt}}{n}\left(r_1\left(xr_1^2 - 1\right) A_2 e^{-r_1 y} + \sqrt{n\beta}\,(xn\beta - 1)\, A A_3 e^{-\sqrt{n\beta}y}\right),\\
J_1(x,y,t) &= \frac{e^{2nt}}{2n}\left(r_1\left(xr_1^2 - 1\right) B_1 e^{-r_1 y} + \sqrt{n\beta}\,(xn\beta - 1)\, B_2 e^{-\sqrt{n\beta}y}\right.\\
&\qquad + r_2\left(xr_2^2 - 1\right) B_3 e^{-r_2 y} + \left(r_1 + \sqrt{n\beta}\right)\left(x\left(r_1 + \sqrt{n\beta}\right)^2\right.\\
&\qquad \left.\left. -1\right)\cdot B_4 e^{-\left(r_1+\sqrt{n\beta}\right)y} + 2\sqrt{n\beta}\,(4xn\beta - 1)\, B B_5 e^{-2\sqrt{n\beta}y}\right).
\end{aligned}
$$

The local skin-friction coefficient, the local heat flux, the local heat transfer, the Nusselt number, the mass flux, the mass transfer coefficient and the Sherwood number are important physical parameters for this type of boundary-layer flow. These parameters can be defined and determined as follows:

The local wall shear stress in the x-direction is given by

$$\tau_\omega(x,t) = (\mu + \mu_r)\left(\frac{\partial u}{\partial y}\right)_{y=0} = x\ell\,(\mu + \mu_r)\left(e^{nt}\tau_{0\omega} + e^{2nt}\tau_{1\omega}\right),$$

where

$$
\begin{aligned}
\tau_{0\omega} &= r_1^2 A_2 + n\beta A A_3,\\
\tau_{1\omega} &= r_1^2 B_1 + n\beta B_2 + r_2^2 B_3 + B_4\left(r_1 + \sqrt{n\beta}\right)^2 + 4n\beta B B_5.
\end{aligned}
$$

The local skin-friction coefficients are defined as

$$C_f = \frac{2\tau_\omega}{\rho\,(\ell x)^2} = 2R_e^{-\frac{1}{2}}\left(e^{nt}\tau_{0\omega} + e^{2nt}\tau_{1\omega}\right),$$
$$C_f R_e^{\frac{1}{2}} = 2\left(e^{nt}\tau_{0\omega} + e^{2nt}\tau_{1\omega}\right).$$

The local heat flux is given by

$$q(x,t) = -\lambda\left(\frac{\partial T}{\partial y}\right)_{y=0} = \lambda T_\omega\left(\frac{l}{v}\right)^{\frac{1}{2}} x^2\left(e^{nt}q_0 + e^{2nt}q_1\right),$$

where

$$\begin{aligned}
q_o &= r_3\Gamma_1 + r_4\Gamma_2,\\
q_1 &= r_5\Gamma_3 + r_6\Gamma_4 + 2\sqrt{n\beta}\widetilde{m}_1 + \widetilde{m}_2(r_1 + \sqrt{n\beta}) + \widetilde{m}_3(r_3 + \sqrt{n\beta})\\
&\quad + \widetilde{m}_4(r_4 + \sqrt{n\beta}) + \widetilde{m}_5(r_1 + r_3) + \widetilde{m}_6(r_1 + r_4) + 2r_1\widetilde{m}_7 + r_3\widetilde{m}_8 + r_4\widetilde{m}_9.
\end{aligned}$$

The heat transfer coefficient may be written as follows:

$$h_\omega(x,t) = \frac{q(x,t)}{T_\omega} = \lambda\left(\frac{l}{v}\right)^{\frac{1}{2}} x^2\left(e^{nt}q_0 + e^{2nt}q_1\right).$$

The local Nusselt number is given by

$$N(x,t) = \frac{h_\omega(x,t)}{\lambda} = \left(\frac{l}{v}\right)^{\frac{1}{2}} x^2\left(e^{nt}q_0 + e^{2nt}q_1\right).$$

The mass flux is given by

$$m_\omega(x,t) = -D\left(\frac{\partial C}{\partial y}\right)_{y=0} = C_\omega\left(\frac{l}{v}\right)^{\frac{1}{2}} x^2\left(e^{nt}m_{0\omega} + e^{2nt}m_{1\omega}\right),$$

where

$$\begin{aligned}
m_{0\omega} &= r_3\ell_1\Gamma_1 + r_4\ell_2\Gamma_2,\\
m_{1\omega} &= r_5\Gamma_3\ell_3 + r_6\Gamma_4\ell_4 + 2\sqrt{n\beta}y\widetilde{m}_{10} + (r_1 + \sqrt{n\beta})\widetilde{m}_{11}\\
&\quad + (r_3 + \sqrt{n\beta})\widetilde{m}_{12} + (r_4 + \sqrt{n\beta})\widetilde{m}_{13} + (r_1 + r_3)\widetilde{m}_{14}\\
&\quad + (r_1 + r_4)\widetilde{m}_{15} + 2r_1\widetilde{m}_{16} + r_3\widetilde{m}_{17} + r_4\widetilde{m}_{18}.
\end{aligned}$$

The mass transfer coefficient may be written as follows:

$$h_m(x,t) = \frac{m_\omega(x,t)}{C_\omega} = D\left(\frac{l}{v}\right)^{\frac{1}{2}} x^2\left(e^{nt}m_{0\omega} + e^{2nt}m_{1\omega}\right).$$

The local Sherwood number is given by

$$Sh_x = \frac{h_m}{D} = \left(\frac{l}{v}\right)^{\frac{1}{2}} x^2\left(e^{nt}m_{0\omega} + e^{2nt}m_{1\omega}\right).$$

5 Results and discussion

The velocity profiles for $\alpha_1 = 0.2$, $\beta = 4.0$, $\alpha = 0.1$, and for different values of f_ω are shown in Fig. 2. As might be expected, suction ($f_\omega < 0$) broadens the velocity distribution increase the thickens of the boundary-layer, while injection ($f_\omega > 0$) make it thins. Also the wall shear stress would be increased with the application of suction whereas injection tends to decrease the wall shear stress. This can be readily understood from the fact that the wall velocity gradient is increased according to the increase of the value of f_ω. The effects of Alfven velocity α on the velocity profiles are presented in Fig. 3 for $f_\omega = 2$, $\alpha_1 = .2$ and $\beta = 4$. It is obvious from this figure that the velocity within the boundary layer decreases as the Alfven velocity increases. In Fig. 4 the transient velocity is plotted for $\alpha = 0.1$, $f_\omega = -2$ and for various values of the parameters α_1 and β. It is seen that, as the value α_1 increase, the velocity of a polar fluid surrounding the surface, increases considerably and it decreases slowly as it goes away from the surface. Also it is observed that any increase in the value of β leads to a decrease in the velocity.

In Fig. 5 the angular velocity profile is plotted to show the effect of material parameters α_1 and β for $\alpha = 0.1$ and $f_\omega = -2$. It is observed that, as well as the value α_1 is increased, the angular velocity of a polar fluid surrounding the surface increases (in magnitude) considerably and it decreases slowly as it goes away from the surface. Also it is observed that any increase

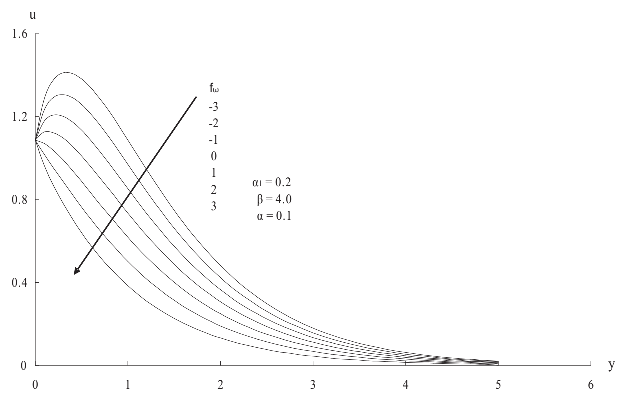

Figure 2: Effect of surface mass transfer on velocity distribution.

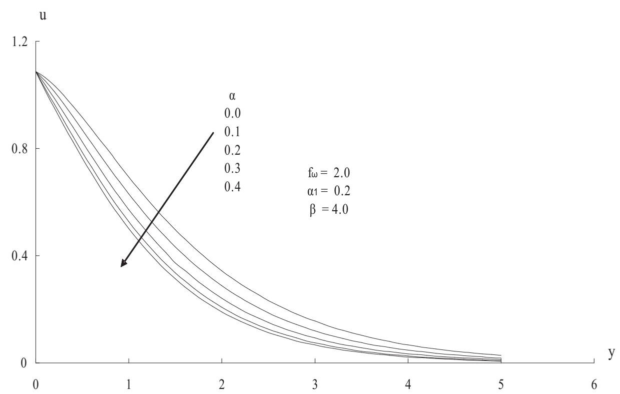

Figure 3: Effect of Alfven velocity α on velocity distribution.

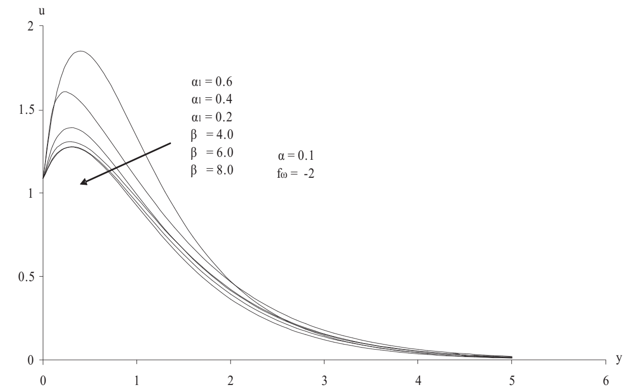

Figure 4: Effect of material parameters α_1 and β on velocity distribution.

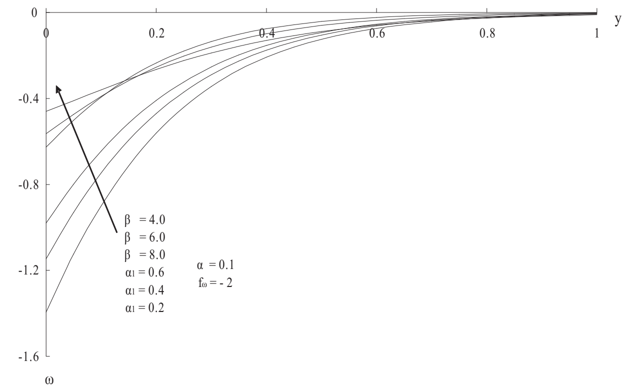

Figure 5: Effect of material parameters α_1 and β on angular velocity distribution.

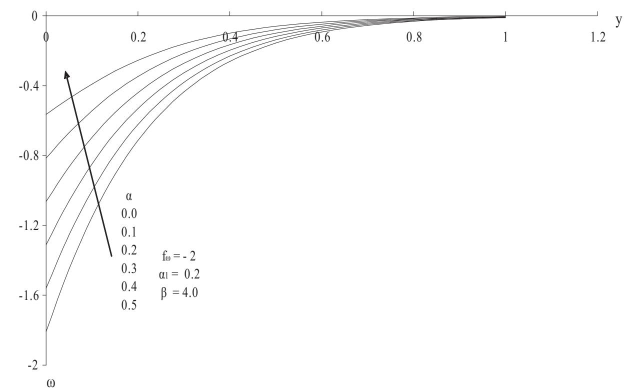

Figure 6: Effect of Alfven velocity α on angular velocity distribution.

in the value of β leads to decrease in the angular velocity (in magnitude). The angular velocity profiles for $\alpha_1 = 0.2$, $\beta = 4.0$, $f_\omega = -2$, and for different values of α are shown in Fig. 6. It is obvious from this figure that the angular velocity within the boundary layer decrease as the Alfven velocity increases. The effects of surface mass transfer on the angular velocity which are displayed in Fig. 7. The effect of mass transfer is to make the angular velocity distribution

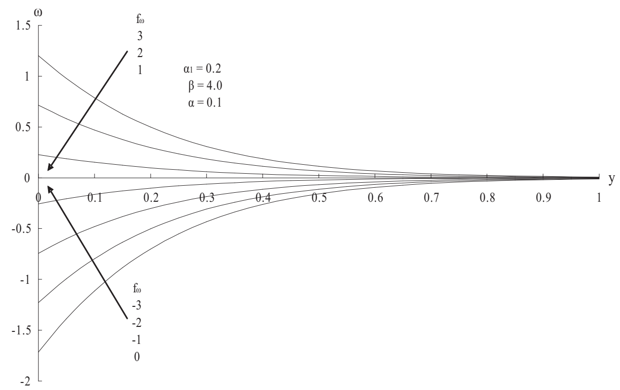

Figure 7: Effect of surface mass transfer on angular velocity distribution.

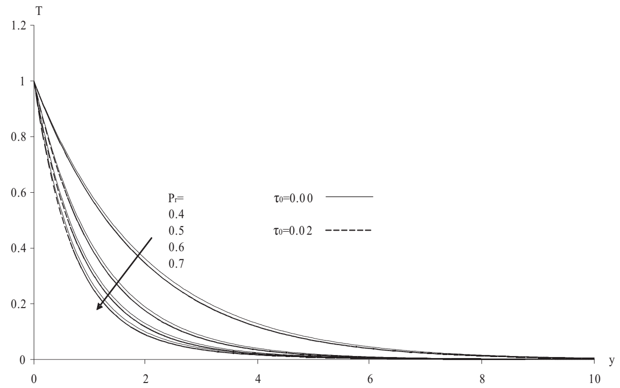

Figure 8: Temperature profiles for various values of rate of Prandtl number P_r.

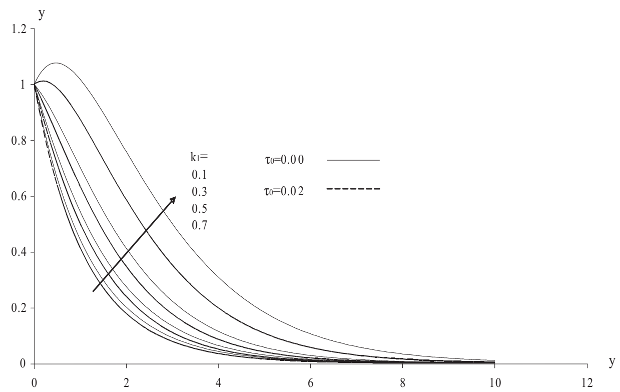

Figure 9: Temperature profiles for various values of rate of specific internal heat generation k_1.

uniform within the boundary layer. In creasing values of the suction leads to an increase in the angular velocity, while an increase in the value of injection leads to an increases in angular velocity (in magnitude).

Results in typical temperature profile are illustrated in Figs. 8 and 9 for various values of Prandtl number, rate of specific internal heat generation (thermal radiation parameter) and

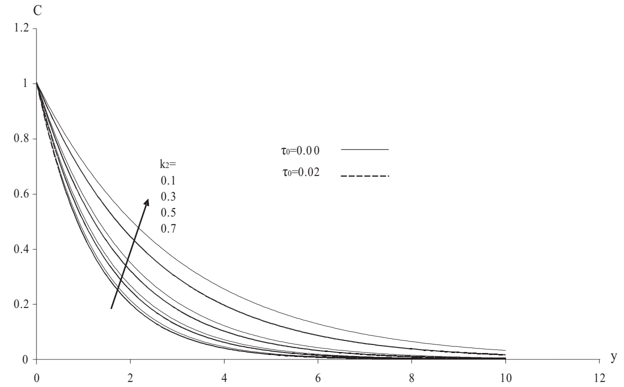

Figure 10: Concentration profiles for various values of reaction rate coefficient k_2.

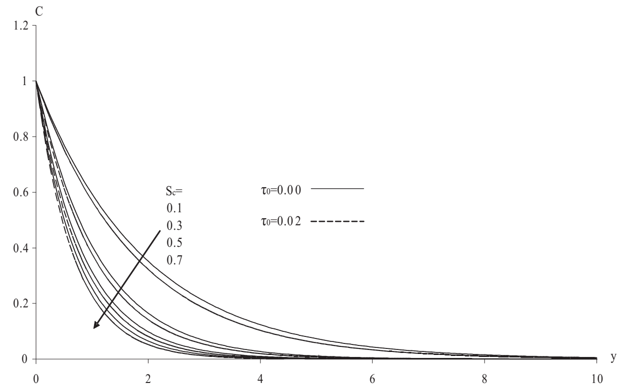

Figure 11: Concentration profiles for various values of Soret number S_c.

relaxation time. The thermal boundary layer thickness is more reduced together with a larger wall temperature gradient when the relaxation time $\tau_0 = 0.02$. Also, it is observed that increases in the value of P_r leads to a decrease in the temperature and any increase in the value of k_1 leads to a decrease in the temperature field.

Figs. 10 and 11 present typical profile for the concentration of different values of the first order of reaction (chemically reactive) k_2 and Schmidt number S_c. The curves illustrate that the concentration slightly increases as k_2 increases and decreases as Schmidt number increases.

6 Conclusions

In all previous studies the combined effects of mass, radiation heat absorption, Dufour, Schmidt, Soret numbers in addition to magnetic field have not been considered simultaneously. In this paper, the effect of radiation heat absorption and mass transfer, Dufour, Schmidt, Soret numbers and Alfven velocity on the flow of a polar fluid in the presence of a induced magnetic and electric fields are taken in consideration since in astrophysical environment the effect of radiation cannot be neglected.

Many metallic materials are manufactured after they have been refined sufficiently in the molten state. Therefore, it is a central problem in metallurgical chemistry to the study of the heat transfer on liquid metal which is a perfect electric conductor. For instance, liquid sodium N a (100° C) and liquid potassium K (100° C) exhibit a very small electrical receptivity ($\rho_L(\exp) = 9.6 \times 10^{-6}\Omega$.cm.) and ($\rho_L(\exp) = 12.97 \times 10^{-6}\Omega$.cm.) [22].

References

[1] J. W. HOYT AND A. G. FABULA, The Effect of Additives on Fluid Friction, U. S. Naval Ordinance Test Station Report, (1964).

[2] W. M. VOGEL AND A. M. PATTERSON, An Experimental Investigation of Additives Injected into the Boundary Layer of an Underwater Body, Pacific Naval Lab. Defense Res. Board of Canada, Report., 2 (1964), pp. 64.

[3] E. L. AERO, A. N. BULYGIN AND E. V. KUVSCHINSKI, *Asymmetric Hydromechanics,* J. Appl. Math. Mech., 29 (1965), pp. 333–346.

[4] N. V. D'EP, *Equations of a Fluid Boundary Layer with Couple Stresses,* J. Appl. Math. Mech., 32 (1968), pp. 777–783.

[5] B. C. SAKIADIS, *Boundary Layer Behavior on Continuous Solid Surfaces: I. Boundary Layer Equations for Two Dimensional and Axis-Symmetric Flow,* AIChE J., 7 (1961), pp. 26–28.

[6] B. C. SAKIADIS, *Boundary Layer Behavior on Continuous Solid Surfaces: II. The Boundary Layer on a Continuous Flat Surface,* AIChE J., 7 (1961), pp. 221–225.

[7] B. C. SAKIADIS, *Boundary Layer Behavior on Continuous Solid Surfaces: III. The Boundary Layer on a Continuous Cylindrical Surface,* AIChE J., 7 (1961), pp. 467–472.

[8] N. T. ELDABE AND M. E. M. OUAF, *Chebyshev Finite Difference Method for Heat and Mass Transfer in a Hydromagnetic Flow of a Micropolar Fluid Past a Stretching Surface with Ohmic Heating and Viscous Dissipation,* Appl. Math. Comput., 177 (2006), pp. 561–571.

[9] N. A. KLESON AND A. DESSEAUX, *Effect of Surface Conditions on Flow of a Micropolar Fluid Driven by a Porous Stretching Sheet,* Int. J. Eng. Sci., 39 (2001), pp. 1881–1897.

[10] I. U. MBELEDOGU AND A. OGULU, *Heat and Mass Transfer of an Unsteady MHD Natural Convection Flow of a Rotating Fluid Past a Vertical Porous Flat Plate in the Presence of Radiative Heat Transfer,* Int. J. Heat Mass Transfer., 50 (2007), pp. 1902–1908.

[11] I-C. LIU, *A Note on Heat and Mass Transfer for a Hydromagnetic Flow over a Stretching Sheet,* Int. J. Heat Mass Transfer., 32 (2005), pp. 1075–1064.

[12] S. K. KHAN, *Heat Transfer in a Viscoelastic Fluid Flow over a Stretching Surface with Heat Source/Link, Suction/Blowing and Radiation,* Int. J. Heat Mass Transfer., 49 (2006), pp. 628–639.

[13] R. CORTELL, *A Note on Flow and Heat Transfer of a Viscoelastic Fluid over a Stretching Sheet,* Non-Linear Mech., 41 (2006), pp. 78–85.

[14] M. ZAKARIA, *Magnetohydrodynamic Unsteady Free Convection Flow of a Couple Stress Fluid with One Relaxation Time through a Porous Medium,* J. Appl. Math. Comp., 146 (2003), pp. 469–494.

[15] M. ZAKARIA, *Problem in Electromagnetic Free Convection Flow of a Micropolar Fluid with Relaxation Time through a Porous Medium,* J. Appl. Math. Comp., 151 (2004), pp. 601–613.

[16] M. EZZAT AND M. ZAKARIA, *Heat Transfer with Thermal Relaxation to a Perfectly Conducting Polar Fluid,* Heat Mass Transfer., 41 (2005), pp. 189–198.

[17] M. EZZAT, M. ZAKARIA AND M. MOURSY, *Magneto-Dydrodynamic Boundary Layer Flow Past a Stretching Plate and Heat Transfer,* J. Appl. Math., 1 (2004), pp. 9–21.

[18] M. EZZAT AND M. ZAKARIA, *Free Convection Effects on a Viscoelastic Boundary Layer Flow with One Relaxation Time Through a Porous Medium,* J. Franklin Inst. B, 334 (1997), pp. 685–706.

[19] M. EZZAT AND M. ZAKARIA, *State Space Approach to Viscoelastic Fluid Flow of Hydromagnetic Fluctuating Boundary-Layer Through a Porous Medium,* Z. Angew. Math. Mech., 77 (1997), pp. 197–207.

[20] Y.Y. LOK, P. PHANG, N. AMIN AND I. POP, *Unsteady Boundary Layer Flow of a Micropolar Fluid near the Forward Stagnation Point of a Plane Surface,* Int. J. Eng. Sci., 41 (2003), pp. 173–186.

[21] M. ZAKARIA, *Thermal Boundary Layer Equation for a Magnetohydrodynamic Flow of a Perfectly Conducting Fluid,* AMC., 148 (2004), pp. 67–79.

[22] A. L. HODGKIN, A. F. HUXLEY AND S. J. ECCLES, Ionic Mechanism Involved in Nerve Cell Activity, The Nobel Prize for Physiology or Medicine, (1963).