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A Family of Methods of the DG-Morley Type for Polyharmonic Equations

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Abstract. Discontinuous Galerkin methods as a solution technique of second order elliptic problems, have been increasingly exploited by several authors in the past ten years. It is generally claimed the alledged attractive geometrical flexibility of these methods, although they involve considerable increase of computational effort, as compared to continuous methods. This work is aimed at proposing a combination of DGM and non-conforming finite element methods to solve elliptic m-harmonic equations in a bounded domain of $\mathbb{R}^n$, for $n = 2$ or $n = 3$, with $m \geq n+1$, as a valid and reasonable alternative to classical finite elements, or even to boundary element methods.

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ for $n = 2$ or $n = 3$, with boundary Γ. For a given $f \in L^2(\Omega)$ we consider the model polyharmonic equation: Find $u \in H_0^m(\Omega)$, such that

$$(-\Delta)^m u = f, \quad \text{for} \quad m \geq 2. \tag{1.1}$$

In the two-dimensional case and for $m = 2$, this equation has several applications in Physics and in Mechanics, while in the three-dimensional case it can be useful in Fluid Mechanics whenever $m = 2$ too (see [7]). As far as the case $m \geq 3$ is concerned, applications of the polyharmonic equation (1.1) were not addressed in the litterature until

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very recently. However in the past few years triharmonic equations have been studied as applied to fluid flow problems [5] or to image processing [11].

If we consider the solution of Eq. (1.1) with conforming finite element methods, functions in the Sobolev spaces $H^m(\Omega)$ for $m \geq 3$ must be approximated by piecewise polynomial functions of the C^{m-1} class. Whenever $n \geq 2$ the construction of such function spaces is a matter of great algebraic complexity. Even in the case where $n = 2$ and $m = 2$ the known constructions are rather elaborated (cf. [2]), let alone the case $m \geq 3$, where the use of such approximation methods becomes unreasonable. This fact naturally leads to external approximations, that is, to the so-called non-conforming methods. In this case the use of polynomials of lower degree is admissible, as long as some conditions are fulfilled in order to ensure the quality of the approximations. More specifically the traces of the polynomials at element interfaces should have suitable continuity properties. Actually for two-dimensional problems a wide spectrum of options of this type has been proposed by several authors since the late sixties, and in this respect we refer to the celebrated Ciarlet's book [2]. For three-dimensional problems only a few non-conforming finite element methods are known for $m = 2$, such as [8]. In the case $n = m = 3$ a classical non-conforming finite element solution method was studied in [9].

Although to date there seems to be little practical use of the m-harmonic equation for high values of m, we address in this work the numerical solution of (1.1), by a method that combines discontinuous Galerkin techniques with classical non-conforming finite elements, for any $m \geq n + 1$. One of the main merits of this method is the fact that it reduces to a minimum the intrinsic complexity of solving the m-harmonic equation in arbitrary domains, even for $m = n + 1$.

In the case $n = 2$ and $m = 3$, a first solution method combining both techniques was proposed in [10]. Here we recall this method as a starting point of a family of methods of this type applying to the case $m \geq n + 1$. As we should say, for two-dimensional problems, the non conforming part of the methodology is aimed at interpolating derivatives of order r with $m - 2 \leq r \leq m - 1$ of the numerical solution, whereas its lower order derivatives and the solution itself are represented by completely discontinuous functions. As a matter of fact, the non conforming part of the approximation method is based on the well-known Morley triangle for solving biharmonic problems (cf. [6]). The idea is extended to the three-dimensional case, in which the non-conforming part is used to interpolate derivatives of order r with $m - 3 \leq r \leq m - 1$, while the lower order derivatives and the function itself are represented by fully discontinuous functions. Here the non-conforming part generalizes the non-conforming tetrahedron introduced in [9] for the case $m = 3$, which in turn are related to the Morley triangle. Indeed it was established in that work that the traces over element interfaces of the cubic functions this finite element is built upon, are nothing but Morley triangles, whenever they happen to be just quadratic. As this property remains valid in our methodology for the natural extension of Morley triangles to the case $m \geq 4$, this explains why we decided to call the new methods a DG-Morley family of methods.

An outline of the paper is as follows. In Section 2 we introduce some notations