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Accurate 8-Node Hybrid Hexahedral Elements with Energy-Compatible Stress Modes

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Abstract. In this paper, an energy-compatibility condition is used for stress optimization in the derivation of new accurate 8-node hexahedral elements for three-dimensional elasticity. Equivalence of the proposed hybrid method to an enhanced strains method is established, which makes it easy to extend the method to general nonlinear problems. Numerical tests show that the resultant elements possess high accuracy at coarse meshes, are insensitive to mesh distortions and free from volume locking in the analysis of beams, plates and shells.

AMS subject classifications: 65N12, 65N30.

Key words: Finite element, hybrid stress method, Hellinger-Reissner principle, locking.

1 Introduction

Due to their computational efficiency and simple geometry, low-order hexahedral and tetrahedral elements are the most exploited in the 3D analysis of general solid and structural mechanics problems. However, conventional low-order elements yield poor results at coarse meshes for problems with bending, and suffer from locking at the nearly incompressible limit. To improve their performance, several kinds of enhanced stress/ strain methods have been developed based on generalized variational principles. The first kind is the assumed stress method based on the Hellinger-Reissner principle, where the displacement and stress fields are the assumed independent variables. Representative of this approach for 2D and 3D analysis are the works [1–25]. The second is the enhanced strain method based on the Hu-Washizu principle, where the displacement, stress and strain fields are the assumed independent variables. In this direction, there are a number of works, e.g., [26–55]. The combined hybrid method is the third kind of enhanced stress method which includes displacement and stress

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variables. It is based on weighted combination of the Hellinger-Reissner functional and its dual, the primal hybrid functional. For the related work one can see [56–59].

Among these methods, the 3-field enhanced strain approach is more attractive in non-linear analysis, since the Hu-Washizu functional is formulated in terms of the strain energy function. But from a practical viewpoint, the 2-field assumed stress approach is more computationally efficient, which makes it more popular. For this method, the choice of stress mode is key to the construction of high performance elements. To improve the performance of the 8-node isoparametric trilinear hexahedron element $H8$ by the assumed stress approach, Spilker and Singh [11] derived an isoparametric quadratic displacement element. However, as the chosen stress field must satisfy equilibrium, then the interpolation must be given in terms of Cartesian co-ordinates. Furthermore, it is necessary to perform inversion operations on a fairly high-order matrix in computing the stiffness matrix. On the other hand, Pian and Tong [2] used isoparametric interpolation and relaxed the equilibrium conditions by introducing Wilson internal displacements parameters [60]. In this way, they constructed a 8-node hybrid stress hexahedral element $PT18\beta$ in which the stress interpolation functions are similar to those given by Loikkanen and Irons [61]. By using admissible matrix formulation, Sze [15–17] improved $PT18\beta$ to obtain better performance for thin plates and shells.

In [23, 25], Xie and Zhou showed that for 2D analysis, fulfillment of the following energy-compatibility (or energy-orthogonality) condition

$$\int_K \boldsymbol{\tau} \cdot \epsilon(\mathbf{v}_I) d\Omega = 0, \qquad \forall \boldsymbol{\tau}, \quad \text{and} \quad \forall \mathbf{v}_I,$$

can lead to optimal stress mode and robust hybrid stress element, where K denotes an arbitrary quadrilateral, $\boldsymbol{\tau}$ the assumed stresses, $\mathbf{v}_I$ the Wilson internal displacements, $\mathbf{n}$ the unit outer normal vector along ∂K. Following the same idea, in this contribution we will use the above stress optimization condition to derive new 8-node hybrid stress hexahedral elements for the analysis of solid mechanical problems, including beams, plates and shells. Following the idea of Piltner [33,34], we will also discuss the equivalence of the new method to an enhanced strains method.

2 Mixed/ hybrid finite element formulations

Consider the linear elasticity problem

$$\begin{cases} -\mathbf{div}\sigma = \mathbf{f}, & \sigma = D\varepsilon(\mathbf{u}), \qquad \text{in} \quad \Omega, \\ \sigma \cdot \mathbf{n}|_{\Gamma_1} = T, & \mathbf{u}|_{\Gamma_0} = 0, \qquad \text{on} \quad \partial\Omega = \Gamma_1 \bigcup \Gamma_0, \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^3$ is a bounded open set, $\mathbf{u}$ represent the displacements, $\boldsymbol{\sigma}$ the stress tensor, $\epsilon(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^T \mathbf{u})/2$ the strains, $\mathbf{D}$ the elasticity module matrix, $\mathbf{f}$ the prescribed body forces, $\mathbf{T}$ the prescribed surface traction on Neumman boundary Γ_1.