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Analysis and Numerical Approximation of an Electro-Elastic Frictional Contact Problem

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Abstract. We consider a mathematical model which describes the static frictional contact between a piezoelectric body and a conductive foundation. A non linear electro-elastic constitutive law is used to model the piezoelectric material. The unilateral contact is modelled using the Signorini condition, nonlocal Coulomb friction law with slip dependent friction coefficient and a regularized electrical conductivity condition. Existence and uniqueness of a weak solution is established. A finite elements approximation of the problem is presented, a priori error estimates of the solutions are derived and a convergent successive iteration technique is proposed.

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1 Introduction

The piezoelectric effect has important uses in modern engineering because it expresses the relation between the electrical and mechanical fields. The effect known as piezoelectricity was discovered by brothers Pierre and Jacques Curie in 1880. They found out when a mechanical stress was applied on some crystals, electrical charges appeared and conversely, the production of stress or strain when an electric field is applied. The piezoelectric materials can be divided in two main groups : crystals and ceramics. The most well-known piezoelectric material is quartz SiO2, also ceramics (BaTiO3, KNbO3, LiNbO3, etc.). General models for elastic materials with piezoelectric effects can be found in [16–18, 22, 23] and, more recently, in [4, 10, 21].

Currently, there is a considerable interest in the study of contact problems involving piezoelectric materials. Thus, static frictional contact problems for electro-elastic

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materials were studied in [2,5,13,14,20], under the assumption that the foundation is insulated, and in [15] under the assumption that the foundation is electrically conductive. Exemple of quasistatic contact model in which the foundation is supposed to be conductive was investigated in [3,12].

In this paper we investigate a mathematical model which describes the static frictional contact between a piezoelectric body and a foundation. The body is supposed to be electro-elastic, with a non-linear elasticity operator. Unlike the models considered in [14,15,20], in the present paper we assume that the contact is modelled using the Signorini condition, nonlocal Coulomb friction law with slip dependent friction coefficient and a regularized electrical conductivity condition, taking into account the conductivity of the foundation as in [12], which involve a coupling between the mechanical and the electrical unknowns. This situation leads to a variational problem which is in form of a coupled system of quasi-variational inequality and non-linear variational equation. To our knowledge, this model has not been studied yet and no result has been obtained for this type problem. We establish the existence and uniqueness of weak solution to this model. Inspired from [8,11], we define the finite elements approximation of the problem and derive the error estimates on the solutions. Then, we introduce an iterative method to solve the nonlinear contact problem, which converges under certain assumptions. An important continuation of this paper consists in the numerical analysis of the model, including numerical simulations will be presented in a forthcoming work.

The paper is structured as follows. In Section 2 we present the model of equilibrium process of the elastic piezoelectric body in frictional contact with a conductive foundation. In Section 3 we introduce the functional spaces for various quantities, list the assumptions on given data and derive the weak formulation of the problem. Then, in Section 4 we state and prove our main existence and uniqueness result, Theorem 4.1. The proofs of these theorems are carried out in several steps and are based on an abstract result in the study of elliptic variational inequalities and Schauder fixed point technique. Finally, in Section 5 we study the finite element approximation of the variational formulation of problems. We prove C`ea´s type inequalities, from which we can conclude the convergence of the finite element method and derive order error estimates under appropriate regularity assumptions on the solution. We introduce an iterative method to solve the resulting finite element system, which converges under certain assumptions.

2 Problem statement

Let

$$\Omega \subset \mathbb{R}^d, \quad d = 2, 3,$$

be the reference domain occupied by the electro-elastic body which is supposed to be open, bounded, with a sufficiently regular boundary $\partial\Omega = \Gamma$. In the sequel we decompose Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand, such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. We assume that the body is fixed on Γ_1 where the displacement field vanishes. The body is acted upon by a volume force of density f_0 and volume electric charges of density q_0 on Ω and a surface traction of density f_2 on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in unilateral contact with friction with a conductive obstacle, the so-called foundation. we model the contact with the Signorini condition and friction. The indices i, j, k, l run between 1 and d. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g.,

$$u_{i,j} = \partial u_i / \partial x_j.$$

Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is $\forall u, v \in \mathbb{R}^d$, $\forall \sigma, \tau \in \mathbb{S}^d$,

$$u \cdot v = u_i \cdot v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}.$$

We shall adopt the usual notations for normal and tangential components of displacement vector and stress,

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu,$$
$$\sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Here and below ν denote the outward normal vector on Γ. Moreover, we denote by

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\sigma = (\sigma_{ij,j}), \quad \text{and} \quad \mathrm{div}\,D = (D_{j,j}),$$

the small strain tensor, Div and div the divergence operator for tensor and vector valued functions, respectively.

Under these conditions the classical formulation of the mechanical problem is as follows:

Problem P. *Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}^d$, an electric potentiel $\varphi : \Omega \to \mathbb{R}$, and an electric displacement field $D : \Omega \to \mathbb{R}^d$, such that*

$$\sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^* E(\varphi), \quad \text{in } \Omega, \tag{2.1}$$
$$D = \mathcal{E}\varepsilon(u) + \beta E(\varphi), \quad \text{in } \Omega, \tag{2.2}$$
$$\mathrm{Div}\,\sigma + f_0 = 0, \quad \text{in } \Omega, \tag{2.3}$$
$$\mathrm{div}\,D = q_0, \quad \text{in } \Omega, \tag{2.4}$$
$$u = 0, \quad \text{on } \Gamma_1, \tag{2.5}$$
$$\sigma\nu = f_2, \quad \text{on } \Gamma_2, \tag{2.6}$$
$$\sigma_\nu(u) \le 0, \ u_\nu \le 0, \ \sigma_\nu(u) u_\nu = 0, \quad \text{on } \Gamma_3, \tag{2.7}$$

$$\|\sigma_\tau\| \le \mu(\|u_\tau\|)|R\sigma_\nu(u)|, \qquad \text{on } \Gamma_3, \tag{2.8a}$$
$$\|\sigma_\tau\| < \mu(\|u_\tau\|)|R\sigma_\nu(u)| \Rightarrow u_\tau = 0, \qquad \text{on } \Gamma_3, \tag{2.8b}$$
$$\|\sigma_\tau\| = -\mu(\|u_\tau\|)|R\sigma_\nu(u)|\frac{u_\tau}{\|u_\tau\|} \Rightarrow u_\tau \neq 0, \qquad \text{on } \Gamma_3, \tag{2.8c}$$
$$\varphi = 0, \qquad \text{on } \Gamma_a, \tag{2.9}$$
$$D \cdot \nu = q_2, \qquad \text{on } \Gamma_b, \tag{2.10}$$
$$D \cdot \nu = \psi(u_\nu)\phi_L(\varphi - \varphi_0), \qquad \text{on } \Gamma_3. \tag{2.11}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$. Eqs. (2.1) and (2.2) represent the electro-elastic constitutive law of the material, in which $\mathfrak{F}$ is a given nonlinear function, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transpose and β denote the electric permittivity tensor. Eqs. (2.3) and (2.4) represent the equilibrium equations for the stress and electric displacement fields, respectively. Relations (2.5) and (2.6) are the displacement and traction boundary conditions, respectively, and (2.9), (2.10) represent the electric boundary conditions. The unilateral boundary conditions (2.7) represent the Signorini law and (2.8) represent the Coulomb's friction law in which μ is the coefficient of friction and R is a regularization operator. Finally, (2.11) represent the regularization electrical contact condition on Γ_3, which was considered in [12], where ϕ_L is the truncation function

$$\phi_L(s) = \begin{cases} -L, & \text{if } \ s < -L, \\ s, & \text{if } \ -L \le s \le L, \\ L, & \text{if } \ s > L, \end{cases}$$

here L is a large positive constant.

In next section we derive the variational formulation of the problem P.

3 Weak formulation

In this section we introduce the notation and recall some definitions needed in the sequel, we introduce the following functional spaces

$$H = L^2(\Omega)^d, \qquad \mathcal{H} = \{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H_1 = H^1(\Omega)^d, \qquad \mathcal{H}_1 = \{\sigma \in \mathcal{H};\ \mathrm{Div}\,\sigma \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \qquad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,$$
$$(u,v)_{H_1} = (u,v)_H + \big(\varepsilon(u),\varepsilon(v)\big)_{\mathcal{H}}, \qquad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\mathrm{Div}\,\sigma, \mathrm{Div}\,\tau)_{\mathcal{H}},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$, and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$, and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v to denote the trace γv of v on Γ.

Let $H_\Gamma^{'}$ be the dual of H_Γ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H_\Gamma^{'}$ and H_Γ. For every $\sigma \in \mathcal{H}_1$, $\sigma\nu$ can be defined as the element in $H_\Gamma^{'}$ which satisfies

$$\langle \sigma\nu, \gamma v\rangle = \big(\sigma, \varepsilon(v)\big)_{\mathcal{H}} + (\mathrm{Div}\, \sigma, v)_H, \qquad \forall v \in H_1. \tag{3.1}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle = \int_\Gamma \sigma\nu \cdot v\, da, \tag{3.2}$$

for all $v \in H_1$, where da is the surface measure element. Keeping in mind the boundary condition (2.5), we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0, \quad \text{on} \quad \Gamma_1\},$$

and K be the set of admissible displacements

$$K = \{v \in V;\ v_\nu \le 0, \quad \text{on} \quad \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$ and Korn's inequality (see [19]) holds,

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_k \|v\|_{H_1}, \qquad \forall v \in V, \tag{3.3}$$

where $c_k > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(u,v)_V = \big(\varepsilon(u), \varepsilon(v)\big)_{\mathcal{H}}, \qquad \|u\|_V = (u,u)_V^{\frac{1}{2}}, \tag{3.4}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.3) that the norms $\|\cdot\|_{H_1}$and $\|\cdot\|_V$ are equivalent on V. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (3.3) and (3.4), there exists a constant $c_0 > 0$ which only depends on the domain Ω, Γ_3 and Γ_1 such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0 \|v\|_V, \qquad \forall v \in V. \tag{3.5}$$

We also introduce the spaces

$$\begin{aligned} &W = \{\psi \in H^1(\Omega)/\psi = 0, \quad \text{on} \quad \Gamma_a\},\\ &\mathcal{W} = \{D = (D_i) \in H^1(\Omega)/(D_i) \in L^2(\Omega), \quad \mathrm{div}\, D \in L^2(\Omega)\}. \end{aligned}$$

The spaces W and $\mathcal{W}$ are reel Hilbert spaces with the inner products

$$\begin{aligned} &(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)},\\ &(D, E)_{\mathcal{W}} = (D, E)_{L^2(\Omega)^d} + (\mathrm{div}\, D, \mathrm{div}\, E)_{L^2(\Omega)}. \end{aligned}$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincar inequality holds

$$\|\nabla\psi\|_{\mathcal{W}} \geq c_F\|\psi\|_W, \qquad \forall\psi \in W, \tag{3.6}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. Moreover, by the Sobolev trace theorem, there exists a constant c_1, depending only on Ω, Γ_a and Γ_3, such that

$$\|\xi\|_{L^2(\Gamma_3)} \leq c_1\|\xi\|_W, \qquad \forall\xi \in W. \tag{3.7}$$

When $D \in \mathcal{W}$ is a sufficiently regular function, the following Green's type formula holds,

$$(D, \nabla\xi)_{L^2(\Omega)^d} + (\operatorname{div} D, \xi)_{L^2(\Omega)} = \int_\Gamma D\cdot\nu\xi\, da, \qquad \forall\xi \in H^1(\Omega). \tag{3.8}$$

Recall also that the transposite $\mathcal{E}^*$ is given by

$$\mathcal{E}^* = (e^*_{ijk}), \qquad \text{where} \quad e^*_{ijk} = e_{kij}, \tag{3.9a}$$

$$\mathcal{E}\sigma v = \sigma\mathcal{E}^* v, \quad \forall\sigma \in \mathbb{S}^d, \ \ v \in \mathbb{R}^d. \tag{3.9b}$$

In the study of mechanical problem (2.1)-(2.11), we assume that the elasticity operator $\mathfrak{F}$ satisfy the following conditions :

$$\begin{cases} \text{(a)} & \mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d, \\ \text{(b)} & \text{There exists } M_{\mathfrak{F}} > 0, \text{ such that} \\ & \|\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2)\| \leq M_{\mathfrak{F}}\|\xi_1 - \xi_2\|, \quad \forall\xi_1, \xi_2 \in \mathbb{S}^d, \quad \text{a.e. } x \in \Omega, \\ \text{(c)} & \text{There exists } m_{\mathfrak{F}} > 0, \text{ such that} \\ & \big(\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2)\big)(\xi_1 - \xi_2) \geq m_{\mathfrak{F}}\|\xi_1 - \xi_2\|^2, \\ & \forall\xi_1, \xi_2 \in \mathbb{S}^d, \quad \text{a.e. } x \in \Omega, \\ \text{(d)} & \text{The mapping } x \to \mathfrak{F}(x,\xi) \text{ is Lebesgue measurable on } \Omega, \ \forall\xi \in \mathbb{S}^d, \\ \text{(e)} & \text{The mapping } x \to \mathfrak{F}(x,0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.10}$$

We note that the condition (3.10) is satisfied in the case of the linear electro-elastic constitutive law , $\sigma = \mathfrak{F}\varepsilon(u) - \mathcal{E}^* E(\varphi)$, in which

$$\mathfrak{F}\,\xi = (f_{ijkl}\xi_{kl}),$$

provided that $f_{ijkl} \in L^\infty(\Omega)$, and there exists $\alpha > 0$, such that

$$f_{ijkl}(x)\xi_k\xi_l \geq \alpha\|\xi\|^2, \qquad \forall\xi \in \mathbb{S}^d, \quad \text{a.e. } x \in \Omega.$$

Exemple of nonlinear operator $\mathfrak{F}$ which satisfy condition (3.10) can be found in [20].

The piezoelectric tensor $\mathcal{E}$ and the electric permittivity tensor β satisfy

$$\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d, \tag{3.11a}$$

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega). \tag{3.11b}$$

$$\begin{cases} (a) & \beta=(\beta_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d,\\ (b) & \beta_{ij}=\beta_{ji}\in L^\infty(\Omega),\\ (c) & \text{there exists } m_\beta>0, \text{ such that } \beta_{ij}E_iE_j\geq m_\beta\|E\|^2,\\ & \forall E\in R^d,\quad \text{a.e. } x\in\Omega. \end{cases} \tag{3.12}$$

The surface electrical conductivity function ψ satisfies

$$\begin{cases} (a) & \psi:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+,\\ (b) & \text{there exists } L_\psi>0, \text{ such that}\\ & |\psi(x,u_1)-\psi(x,u_2)|\leq L_\psi|u_1-u_2|,\ \forall u_1,u_2\in\mathbb{R},\quad \text{a.e. } x\in\Gamma_3,\\ (c) & \text{there exists } M_\psi>0, \text{ such that}\\ & |\psi(x,u)|\leq M_\psi,\ \forall u\in\mathbb{R},\quad \text{a.e. } x\in\Gamma_3,\\ (d) & x\to\psi(x,u) \text{ is mesurable on } \Gamma_3, \text{for all } u\in\mathbb{R},\\ (e) & x\to\psi(x,u)=0,\ \text{ for all } u\leq 0. \end{cases} \tag{3.13}$$

The coefficient of friction satisfies

$$\begin{cases} (a) & \mu:\Gamma_3\times\mathbb{R}_+\to\mathbb{R}_+,\\ (b) & \text{there exists } L_\mu>0, \text{ such that}\\ & |\mu(.,u)-\mu(.,v)|\leq L_\mu|u-v|,\ \forall u,v\in\mathbb{R}_+,\quad \text{a.e. } x\in\Gamma_3,\\ (c) & \exists\mu^*>0, \text{ such that } \mu(x,u)\leq\mu^*,\ \forall u\in\mathbb{R}_+,\quad \text{a.e. } x\in\Gamma_3,\\ (d) & \text{the function } x\to\mu(x,u) \text{ is measurable on } \Gamma_3, \text{for all } u\in\mathbb{R}+. \end{cases} \tag{3.14}$$

We assume that the body forces, the tractions, the volume and surface charge densities satisfy

$$f_0\in L^2(\Omega)^d,\qquad f_2\in L^2(\Gamma_3)^d, \tag{3.15a}$$

$$q_0\in L^2(\Omega),\qquad q_2\in L^2(\Gamma_b). \tag{3.15b}$$

Also, the given potential is such that

$$\varphi_0\in L^2(\Gamma_3). \tag{3.16}$$

Next, we use Riesz's representation theorem, consider the elements $f\in V$, and $q\in W$ given by

$$(f,v)_V=\int_\Omega f_0\cdot v\,dx+\int_{\Gamma_2}f_2\cdot v da,\quad \forall v\in V, \tag{3.17a}$$

$$(q,\xi)_W=\int_\Omega q_0\xi\,dx-\int_{\Gamma_b}q_2\xi da,\qquad \forall\xi\in W, \tag{3.17b}$$

and we define the mappings $j:V\times V\to\mathbb{R}$, and $\ell:V\times W\times W\to\mathbb{R}$, respectively, by

$$\ell(u,\varphi,\xi)=\int_{\Gamma_3}\psi(u_\nu)\phi_L(\varphi-\varphi_0)\xi da,\qquad \forall u\in V,\ \forall\varphi,\xi\in W, \tag{3.18a}$$

$$j(u,v)=\int_{\Gamma_3}\mu(\|u_\tau\|)|R\sigma_\nu(u)|\|v_\tau\,da,\qquad \forall u,v\in V. \tag{3.18b}$$

Keeping in mind assumptions (3.13)-(3.16), it follows that the integrals in (3.17a)-(3.18b) are well-defined. Finally, we assume that R: $H^{'}_{\Gamma_3} \to L^\infty(\Gamma_3)$, is a linear and continuous mapping (see [24]). Using Grenn's formula (3.1), (3.2) and (3.8), it is straightforward to see that if $(u,\ \sigma,\ \varphi,\ D)$ are sufficiently regular function, which satisfy (2.3)-(2.11) then

$$\big(\sigma, \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + j(u,v) - j(u,u) \geq (f, v-u)_V, \quad \forall v \in K, \tag{3.19a}$$

$$(D, \nabla\xi)_{L^2(\Omega)^d} = \ell(u,\varphi,\xi) - (q,\xi)_W, \qquad \forall \xi \in W. \tag{3.19b}$$

We plug (2.1) in (3.19a), (2.2) in (3.19b), and use the notation $E{=}-\nabla\varphi$ to obtain the following variational formulation of Problem P, in the terms of displacement field and electric potential.

Problem PV. *Find a displacement field $u \in K$, and an electric potential $\varphi \in W$, such that*

$$\begin{aligned} &\big(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + \big(\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u)\big)_{L^2(\Omega)^d} + j(u,v) - j(u,u) \\ \geq&(f, v-u)_V, \quad \forall v \in K, \end{aligned} \tag{3.20a}$$

$$\begin{aligned} &(\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} - \big(\mathcal{E}\varepsilon(u), \nabla\xi\big)_{L^2(\Omega)^d} + \ell(u,\varphi,\xi) \\ =&(q,\xi)_W, \qquad \forall \xi \in W. \end{aligned} \tag{3.20b}$$

4 Existence and uniqueness

The main existence and uniqueness result, which we establish in this section, is the following.

Theorem 4.1. *Assume that* (3.10)-(3.12), (3.13)[(a), (c), (d), (e)], (3.14)[(a), (c), (d)] *and* (3.16) *hold. Then :*

(1) The problem PV has at least one solution $(u,\varphi) \in K \times W$;

(2) Under the assumptions (3.13)(b) *and* (3.14)(b), *there exists $L^* > 0$, such that if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$, then the problem* PV *has a unique solution.*

The proof of Theorem 4.1 will be carried out in several steps. To present it, we consider the product spaces $X = V \times W$, and $Y = L^2(\Gamma_3) \times L^2(\Gamma_3)$, together with the inner products

$$(x,y)_X = (u,v)_V + (\varphi,\xi)_W, \qquad \forall x = (u,\varphi),\ \ y = (v,\xi) \in X, \tag{4.1}$$

$$(\eta,\theta)_Y = (g,\lambda)_{L^2(\Gamma_3)} + (z,\zeta)_{L^2(\Gamma_3)}, \qquad \forall \eta = (g,z),\ \ \theta = (\lambda,\zeta) \in Y, \tag{4.2}$$

and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $U = K \times W$ be non-empty closed convex subset of X. We define the operator $A : U \to X$, the functions $\widetilde{j}$, $\widetilde{\ell}$ on $U \times X$, and the element $f_3 \in X$ by equalities :

$$\begin{aligned} (Ax,y)_X = &\big(\mathfrak{F}\varepsilon(u), \varepsilon(v)\big)_{\mathcal{H}} + (\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} + \big(\mathcal{E}^*\nabla\varphi, \varepsilon(v)\big)_{L^2(\Omega)^d} \\ &- (\mathcal{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d}, \quad \forall x = (u,\varphi),\ \ y = (v,\xi) \in U, \end{aligned} \tag{4.3}$$

$$\widetilde{j}(x,y) = j(u,v), \qquad \forall x = (u,\varphi),\ \ y = (v,\xi) \in X, \tag{4.4}$$

$$\widetilde{\ell}(x,y) = \int_{\Gamma_3} \psi(u_\nu)\phi_L(\varphi - \varphi_0)\xi d\gamma, \quad \forall x = (u,\varphi),\ \ y = (v,\xi) \in X, \tag{4.5}$$

$$f_3 = (f,q) \in X. \tag{4.6}$$

We start the following equivalence result

Lemma 4.1. *The couple $x = (u,\varphi)$ is a solution to problem PV if and only if*

$$(Ax, y - x)_X + \widetilde{j}(x,y) - \widetilde{j}(x,x) + \widetilde{\ell}(x, y - x) \geq (f_3, y - x)_X, \tag{4.7}$$

for all $y = (v,\xi) \in K \times W$.

Proof. Let $x = (u,\varphi) \in U$ be a solution to problem PV and let $y = (v,\xi) \in U$. We use the test function $\xi - \varphi$ in (3.20b), add the corresponding inequality to (3.20a) and use (4.1) and (4.3)-(4.6) to obtain (4.7). Conversely, let $x = (u,\varphi) \in U$ be a solution to the elliptic variational inequalities (4.7). We take $y = (v,\varphi)$ in (4.7), where v is an arbitrary element of K and obtain (3.20a). Then for any $\xi \in W$, we take successively $y = (v, \varphi + \xi)$, and $y = (v, \varphi - \xi)$ in (4.7) to obtain (3.20b), which concludes the proof of lemma 4.1. ☐

We define two closed convex set

$$\mathcal{K}_1 = \{g \in L^2(\Gamma_3)/g \geq 0, \ \text{ and } \ \|g\|_{L^2(\Gamma_3)} \leq k_1\},$$
$$\mathcal{K}_2 = \{z \in L^2(\Gamma_3)/\|z\|_{L^2(\Gamma_3)} \leq k_2\},$$

with k_1 and k_2 to be specified, and we define the functions

$$\ell_z(\xi) = \int_{\Gamma_3} z\xi \, da, \qquad \forall \xi \in W, \tag{4.8}$$

$$j_g(v) = \int_{\Gamma_3} g \, \|v_\tau\| \, da, \quad \forall v \in K. \tag{4.9}$$

Let $\eta = (g,z) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$ be given and consider q_3 such that

$$(q_3,\xi)_W = (q,\xi)_W - \ell_z(\xi), \tag{4.10}$$

for all $\xi \in W$, and note that (3.17b) and (4.8) imply that $q_3 \in W$. We consider the element $f_\eta \in X$ given by

$$f_\eta = (f, q_3) \in X. \tag{4.11}$$

We extend the functional j_g defined by (4.9) to a functional $\widetilde{j_g}$ defined on U, that is

$$\widetilde{j_g}(x) = j_g(u), \quad \forall x = (u,\varphi) \in U, \tag{4.12}$$

and consider the following intermediate problem

$$(Ax_\eta, y - x_\eta)_X + \widetilde{j_g}(y) - \widetilde{j_g}(x_\eta) \geq (f_\eta, y - x_\eta)_X, \quad \forall y = (v,\xi) \in U. \tag{4.13}$$

We have the following existence and uniqueness result.

Lemma 4.2. *For any $\eta \in \mathcal{K}_1 \times \mathcal{K}_2$, assume that* (3.10)-(3.12) *hold. Then*

(i) The problem (4.13) *has a unique solution $x_\eta=(u_\eta,\varphi_\eta)\in K\times W$, which depends Lipschitz continuously on $\eta \in L^2(\Gamma_3)\times L^2(\Gamma_3)$,*

(ii) There exists a constant $c_2>0$, such that the solution of problem (4.13) *satisfies*

$$\|x_\eta\|_X \le c_2\|f_\eta\|_X. \tag{4.14}$$

Proof. The proof of lemma 4.2 is based on the following abstract result for elliptic variational inequalities (see [6]).

Theorem 4.2. *Let X be a Hilbert space, and $U\subset X$ be a nonempty, convex and closed subset. Assume that $A:U\to X$ is a strongly monotone and Lipschitz continuous operator on X, i.e.,*

$$\begin{aligned} &\exists m>0, \qquad (Au-Av,u-v)_X \ge m\|u-v\|_X^2,\\ &\exists M>0, \qquad \|Au-Av\|_X \le M\|u-v\|_X, \end{aligned}$$

and that $j:U\to(-\infty,\infty]$ is a proper, convex and lower semicontinuous function. Then, for each $f\in X$, the elliptic variational inequality of the second kind, $\forall x\in U$,

$$(Ax,y-x)_X + j(y)-j(x) \ge (f,y-x)_X, \qquad \forall y\in U,$$

has a unique solution.

Let the operator A and the functional $\widetilde{j_g}$ given by (4.3) and (4.12), respectively. In order to use this abstract result, we prove that

(a) The operator $A:X\to X$ is strongly monotone and Lipschitz continuous,

(b) The functional $\widetilde{j_g}$ is proper, convex and continuous.

First, consider two elements $x_1=(u_1,\varphi_1),\ x_2=(u_2,\varphi_2)\in X$, using (4.3), we have

$$\begin{aligned} &(Ax_1-Ax_2,x_1-x_2)_X\\ =&\big(\mathfrak{F}\varepsilon(u_1)-\mathfrak{F}\varepsilon(u_2),\varepsilon(u_1)-\varepsilon(u_2)\big)_{\mathcal{H}} + \big(\beta\nabla\varphi_1-\beta\nabla\varphi_2,\nabla\varphi_1-\nabla\varphi_2\big)_{L^2(\Omega)^d}\\ &+\big(\mathcal{E}^*\nabla\varphi_1-\mathcal{E}^*\nabla\varphi_1,\varepsilon(u_1)-\varepsilon(u_2)\big)_{L^2(\Omega)^d} - \big(\mathcal{E}\varepsilon(u_1)-\mathcal{E}\varepsilon(u_2),\nabla\varphi_1-\nabla\varphi_2\big)_{L^2(\Omega)^d}, \end{aligned}$$

and since if follows by (3.9a) that $\big(\mathcal{E}^*\nabla\varphi,\varepsilon(u)\big)_{\mathcal{H}} = \big(\mathcal{E}\varepsilon(u),\nabla\varphi\big)_{L^2(\Omega)^d}$, for all $x=(u,\varphi)$, we find

$$\begin{aligned} &(Ax_1-Ax_2,x_1-x_2)_X\\ =&\big(\mathfrak{F}\varepsilon(u_1)-\mathfrak{F}\varepsilon(u_2),\varepsilon(u_1)-\varepsilon(u_2)\big)_{\mathcal{H}} + \big(\beta\nabla\varphi_1-\beta\nabla\varphi_2,\nabla\varphi_1-\nabla\varphi_2\big)_{L^2(\Omega)^d}. \end{aligned}$$

We use now (3.10), (3.12) and (3.6), there exists $c_3>0$, which depends only on $\mathfrak{F}$, β, Ω and Γ_a, such that

$$(Ax_1-Ax_2,x_1-x_2)_X \ge c_3\big(\|u_1-u_2\|_V^2 + \|\varphi_1-\varphi_2\|_W^2\big),$$

and keeping in mind (4.1), we obtain

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq c_3 \|x_1 - x_2\|_X^2. \tag{4.15}$$

In the same way, using (3.10)-(3.12), after some algebra, it follows that there exists $c_4 > 0$, which depends only on $\mathfrak{F}$, β and $\mathcal{E}$, such that

$$\begin{aligned}&(Ax_1 - Ax_2, y)_X\\ \leq& c_4\big(\|u_1 - u_2\|_V \|v\|_V + \|\varphi_1 - \varphi_2\|_W \|v\|_V + \|u_1 - u_2\|_V \|\xi\|_W + \|\varphi_1 - \varphi_2\|_W \|\xi\|_W\big),\end{aligned}$$

for all $y = (v, \xi) \in X$. We use (4.1) and the previous inequality to obtain

$$(Ax_1 - Ax_2, y)_X \leq 4c_4(\|x_1 - x_2\|_X \, \|y\|_X), \quad \forall y \in X,$$

and taking $y = Ax_1 - Ax_2 \in X$, we find

$$\|Ax_1 - Ax_2\|_X \leq 4\, c_4 \|x_1 - x_2\|_X, \tag{4.16}$$

(a) is now a consequence of inequalities (4.15) and (4.16). Next, we investigate the properties of the functional $\widetilde{j_g}$ given by (4.12), (4.9). We first remark that $\widetilde{j_g}$ is proper and convex on U. Let $x_1 = (u_1, \varphi_1), x_2 = (u_2, \varphi_2) \in U$, we have

$$\begin{aligned}\big|\widetilde{j_g}(x_1) - \widetilde{j_g}(x_2)\big| &= \Big| \int_{\Gamma_3} g(\|u_{1\tau}\| - \|u_{2\tau}\|) da \Big|\\ \leq& \int_{\Gamma_3} g \|u_{1\tau} - u_{2\tau}\| da\\ \leq& \|g\|_{L^2(\Gamma_3)} \|u_1 - u_2\|_{L^2(\Gamma_3)^d}.\end{aligned}$$

Using (3.5), we obtain

$$\big|\widetilde{j_g}(x_1) - \widetilde{j_g}(x_2)\big| \leq c_0 \, \|g\|_{L^2(\Gamma_3)} \, \|u - v\|_V.$$

Now, by (4.1), we find that

$$\big|\widetilde{j_g}(x_1) - \widetilde{j_g}(x_2)\big| \leq c_0 \, \|g\|_{L^2(\Gamma_3)} \, \|x_1 - x_2\|_X.$$

Thus $\widetilde{j_g}$ is Lipschitz continuous, and therefore, $\widetilde{j_g}$ is a fortiori lower semicontinuous function.

Using (a), (b) and abstract results of Theorem 4.2, we obtain that problem (4.13) has a unique solution. We show next that this solution depends Lipschitz continuously on $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$. Let $\eta_1 = (g_1, z_1)$, $\eta_2 = (g_2, z_2) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$ be given, and denote the corresponding solution of the problem (4.13) by $x_{\eta_1} = (u_{\eta_1}, \varphi_{\eta_1})$, and $x_{\eta_2} = (u_{\eta_2}, \varphi_{\eta_2})$. Then we have

$$\begin{aligned}&(Ax_{\eta_1}, y - x_{\eta_1})_X + \widetilde{j_{g_1}}(y) - \widetilde{j_{g_1}}(x_{\eta_1}) \geq (f_{\eta_1}, y - x_{\eta_1})_X, \quad \forall y \in U,\\ &(Ax_{\eta_2}, y - x_{\eta_2})_X + \widetilde{j_{g_2}}(y) - \widetilde{j_{g_2}}(x_{\eta_2}) \geq (f_{\eta_2}, y - x_{\eta_2})_X, \quad \forall y \in U.\end{aligned}$$

We take $y = x_{\eta_2}$ in the first inequality, and $y = x_{\eta_1}$ in the second inequality, successively, we obtain

$$\begin{aligned}&(Ax_{\eta_1} - Ax_{\eta_2}, x_{\eta_1} - x_{\eta_2})\\ \le& \int_{\Gamma_3} (g_1 - g_2)(\|u_{\eta_1\tau}\| - \|u_{\eta_2\tau}\|)\, da + \int_{\Gamma_3} (z_1 - z_2)(\varphi_{\eta_1} - \varphi_{\eta_2})\, da\\ \le&\|g_1 - g_2\|_{L^2(\Gamma_3)} \|u_{\eta_1\tau} - u_{\eta_2\tau}\|_{L^2(\Gamma_3)^d} + \|z_1 - z_2\|_{L^2(\Gamma_3)} \|\varphi_{\eta_1} - \varphi_{\eta_2}\|_{L^2(\Gamma_3)}.\end{aligned}$$

Thus, using (3.5) and (3.7), we deduce

$$\begin{aligned}&(Ax_{\eta_1} - Ax_{\eta_2}, x_{\eta_1} - x_{\eta_2})_X\\ \le& c_0\|g_1 - g_2\|_{L^2(\Gamma_3)} \|u_{\eta_1} - u_{\eta_2}\|_V + c_1\|z_1 - z_2\|_{L^2(\Gamma_3)} \|\varphi_{\eta_1} - \varphi_{\eta_2}\|_W,\end{aligned}$$

and using (4.1), (4.15), and (4.2)

$$\|x_{\eta_1} - x_{\eta_2}\|_X \le \tfrac{\max\{c_0,c_1\}}{c_3} \sqrt{2}\|(g_1, z_1) - (g_2, z_2)\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)},$$

thus there exists a positive constant $c_5 = \sqrt{2}\max(c_0, c_1)/c_3$, such that

$$\|x_{\eta_1} - x_{\eta_2}\|_X \le c_5\|\eta_1 - \eta_2\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}, \tag{4.17}$$

whence (i) follows. We turn now to the proof of (ii). Let $\eta=(g, z)\in\mathcal{K}_1 \times \mathcal{K}_2$, we take $y = 0$ in the inequality (4.13), we have

$$(Ax_\eta, x_\eta)_X + \widetilde{j_g}(x_\eta) \le (f_\eta, x_\eta)_X, \qquad \forall x_\eta \in X.$$

As $g \ge 0$, we obtain

$$(Ax_\eta, x_\eta)_X \le (f_\eta, x_\eta)_X, \qquad \forall x_\eta \in X,$$

using (4.15), we deduce

$$\|x_\eta\|_X \le \frac{1}{c_3}\|f_\eta\|_X,$$

and the lamma is proved. □

We now consider the operator $\Lambda : L^2(\Gamma_3) \times L^2(\Gamma_3) \to L^2(\Gamma_3) \times L^2(\Gamma_3)$, such that for all $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$, we have

$$\Lambda\eta = \big(\mu(\|u_{\eta\tau}\|)|R\sigma_\nu(u_\eta)|, \psi(u_{\eta\nu})\phi_L(\varphi_\eta - \varphi_0)\big), \qquad \forall \eta \in L^2(\Gamma_3) \times L^2(\Gamma_3),$$

it follows from assumptions (3.13)-(3.14) that the operator Λ is well-defined. In order to prove that Λ has a fixed point, we will need the following result:

Lemma 4.3. *The mapping $\eta \to x_\eta$, where x_η is the solution to* (4.13)*, is weakly continuous from $L^2(\Gamma_3) \times L^2(\Gamma_3)$ to X.*

Proof. Let a sequence $\big(\eta_n = (g_n, z_n)\big)$ in $L^2(\Gamma_3) \times L^2(\Gamma_3)$ converging weakly to $\eta = (g, z)$, we denote by $x_{\eta_n} = (u_{\eta_n}, \varphi_{\eta_n}) \in U$ the solution of (4.13) corresponding to η_n, then we have

$$(Ax_{\eta_n}, y - x_{\eta_n})_X + \widetilde{j_{g_n}}(y) - \widetilde{j_{g_n}}(x_{\eta_n}) \ge (f_{\eta_n}, y - x_{\eta_n})_X, \qquad \forall y = (v, \xi) \in U, \tag{4.18}$$

where

$$(f_{\eta_n}, y - x^n_{\eta_n})_X = (f, v - u_{\eta_n})_V + (q, \xi - \varphi_{\eta_n})_W - h_{z_n}(\xi - \varphi_{\eta_n}),$$

taking $y = 0$ in (4.18) and using (4.15), (3.5) and (4.15), we deduce

$$\begin{aligned}\|x_{\eta_n}\|_X \leq & \frac{1}{c_3}\big(\|f_{\eta_n}\|_X + c_0\|g_n\|_{L^2(\Gamma_3)}\big)\\ \leq & \frac{1}{c_3}\big(\|f\|_V + \|q\|_W + \|z_n\|_{L^2(\Gamma_3)} + c_0\|g_n\|_{L^2(\Gamma_3)}\big)\\ \leq & c\big(\|f\|_V + \|q\|_W + \|\eta_n\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}\big),\end{aligned}$$

that is, the sequence (x_{η_n}) is bounded in X, then, there exists $\widetilde{x}=(\widetilde{u},\widetilde{\varphi})\in X$, and a subsequence, denote again (x_{η_n}), such that

$$x_{\eta_n} \rightharpoonup \widetilde{x} \in X, \quad \text{as } n \to +\infty.$$

Moreover, U is closed convex set in a real Hilbert space X, therefor U is weakly closed, then $\widetilde{x} \in U$.

We next prove that $\widetilde{x}$ is solution of (4.13). First, we prove that

$$(f_{\eta_n}, y - x_{\epsilon\eta_n})_X \to (f_\eta, y - \widetilde{x})_X, \quad \text{as } n \to +\infty. \tag{4.19}$$

We have

$$\begin{aligned}\big|h_{z_n}(\xi - \widetilde{\varphi}) - h_{z_n}(\xi - \varphi_{\eta_n})\big| \leq & \|z_n\|_{L^2(\Gamma_3)}\|\widetilde{\varphi} - \varphi_{\eta_n}\|_{L^2(\Gamma_3)}\\ \leq & \|\underbrace{\eta_n}_{bounded}\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}\|\widetilde{x} - x_{\eta_n}\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}.\end{aligned}$$

Since the trace map $\gamma : X \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$ is compact operator, from the weak convergence $x_{\eta_n} \rightharpoonup \widetilde{x}$ in X, we obtain the convergence $x_{\eta_n} \to \widetilde{x}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$. so we have (4.19).

Now, form (4.18), we have

$$\begin{aligned}&(Ax_{\eta_n}, y - x_{\eta_n})_X\\ \geq & (f_{\eta_n}, y - x^n_{\eta_n})_X - \big(\widetilde{j_{g_n}}(y) - \widetilde{j_{g_n}}(\widetilde{x})\big) - \big(\widetilde{j_{g_n}}(\widetilde{x}) - \widetilde{j_{g_n}}(x^n_{\eta_n})\big), \qquad \forall y = (v, \xi) \in U.\end{aligned}$$

Since

$$\begin{aligned}\big|\widetilde{j_{g_n}}(\widetilde{x}) - \widetilde{j_{g_n}}(x_{\eta_n})\big| \leq & \|g_n\|_{L^2(\Gamma_3)}\|\widetilde{u} - u_{\eta_n}\|_{L^2(\Gamma_3)}\\ \leq & \|\underbrace{\eta_n}_{bounded}\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}\|\widetilde{x} - x_{\eta_n}\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)},\end{aligned}$$

so we obtain

$$\limsup_{n\to+\infty}(Ax_{\eta_n}, x_{\eta_n} - y)_X \leq (f_\eta, \widetilde{x} - y)_X + \big(\widetilde{j_{g_n}}(y) - \widetilde{j_{g_n}}(\widetilde{x})\big), \quad \forall y = (v, \xi) \in U. \tag{4.20}$$

Then, by (4.20), one gets

$$\begin{aligned}&\limsup_{n\to+\infty}(Ax_{\eta_n}, x_{\eta_n}-\widetilde{x})_X\\ =&\limsup_{n\to+\infty}\Big((Ax_{\eta_n}, x_{\eta_n}-y)_X+(Ax_{\eta_n}, y-\widetilde{x})_X\Big)\\ \le&\limsup_{n\to+\infty}\Big((Ax_{\eta_n}, x_{\eta_n}-y)_X+\|Ax_{\eta_n}\|_X\|y-\widetilde{x}\|_X\Big)\\ \le&(f_\eta,\widetilde{x}-y)_X+\big(\widetilde{j_{g_n}}(y)-\widetilde{j_{g_n}}(\widetilde{x})\big)+\limsup_{n\to+\infty}\Big(\|Ax_{\eta_n}\|_X\|y-\widetilde{x}\|_X\Big),\end{aligned}$$

for all $y=(v,\xi)\in U$. Note that $(\|Ax_{\eta_n}\|_X)$ is bounded (according to (3.10)(e) and (4.16)), and we may then substitute $y=\widetilde{x}$ into the previous inequality to obtain

$$\limsup_{n\to+\infty}(Ax_{\eta_n}, x_{\eta_n}-\widetilde{x})_X\le 0.$$

By pseudomonotonicity of A, we get

$$(A\widetilde{x},\widetilde{x}-y)_X\le\liminf_{n\to+\infty}(Ax^n_{\eta_n}, x^n_{\eta_n}-y)_X. \tag{4.21}$$

Combining (4.18), (4.19) and (4.21), one gets

$$\widetilde{x}\in U, \tag{4.22a}$$

$$(A\widetilde{x}, y-\widetilde{x})_X+\widetilde{j}_g(y)-\widetilde{j}_g(\widetilde{x})\ge(f_\eta, y-\widetilde{x})_X,\qquad\forall y=(v,\xi)\in U, \tag{4.22b}$$

from (4.22), we find that $\widetilde{x}$ is a solution of problem (4.13) and from the uniqueness of the solution for this variational inequality we obtain $\widetilde{x}=x_\eta$. Since x_η is the unique weak limit of any subsequence of (x_{η_n}), we deduce that the whole sequence (x_{η_n}) is weakly convergent in X to x_η, ensures that the weak continuous mapping $\eta\to x_\eta$, from $L^2(\Gamma_3)\times L^2(\Gamma_3)$ to X. ☐

Lemma 4.4. *If*

$$k_1=c_2\mu^*c_0c_*\Big(\|f\|_V+\|q\|_W+M_\psi L\,meas(\Gamma_3)^{\frac{1}{2}}\Big),\quad and\quad k_2=M_\psi L\,meas(\Gamma_3)^{\frac{1}{2}},$$

then the operator Λ has at least one fixed point.

Proof. Let $\eta=(g,z)\in\mathcal{K}_1\times\mathcal{K}_2$, i.e.,

$$\|g\|_{L^2(\Gamma_3)}\le k_1,\quad\text{and}\quad\|z\|_{L^2(\Gamma_3)}\le k_2.$$

Keeping in mind (4.2), it follows that

$$\|\eta\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}\le k_1+k_2,$$

from (4.11)-(4.10), we obtain

$$(f_\eta, x_\eta)_X\le(f,u_\eta)_V+(q,\varphi_\eta)_W-\ell_z(\varphi_\eta), \tag{4.23}$$

using (4.23) in (4.14), we deduce

$$\|x_\eta\|_X \le c_2\big(\|f\|_V + \|q\|_W + \|z\|_{L^2(\Gamma_3)}\big). \tag{4.24}$$

Since $z = \psi(u_{\eta\nu})\phi_L(\varphi_\eta - \varphi_0)$, form the bounds $\psi(u_{\eta\nu}) \le M_\psi$, and $\phi_L(\varphi_\eta - \varphi_0) \le L$,

$$\|z\|_{L^2(\Gamma_3)} \le M_\psi L\ meas(\Gamma_3)^{\frac{1}{2}}, \tag{4.25}$$

taking (4.25) in (4.24), we find

$$\|x_\eta\|_X \le c_2\Big(\|f\|_V + \|q\|_W + M_\psi L\ meas(\Gamma_3)^{\frac{1}{2}}\Big). \tag{4.26}$$

From (4.18) and (4.2), we have

$$\big\|\Lambda\eta\big\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)} \le \big\|\mu(\|u_{\eta\tau}\|)|R\sigma_\nu(u_\eta)|\big\|_{L^2(\Gamma_3)} + \big\|\psi(u_{\eta\nu})\phi_L(\varphi_\eta - \varphi_0)\big\|_{L^2(\Gamma_3)},$$

using (3.14)(c), (3.5), (4.25), (4.1), (4.26), and the continuity of R, yield that there exists a constant $c_* > 0$, such that

$$\|\Lambda\eta\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)} \le c_2\mu^* c_0 c_*\Big(\|f\|_V + \|q\|_W + M_\psi L\ meas(\Gamma_3)^{\frac{1}{2}}\Big) + M_\psi L\ meas(\Gamma_3)^{\frac{1}{2}},$$

if one selected for value of

$$k_1 = c_2\mu^* c_0 c_*\Big(\|f\|_V + \|q\|_W + M_\psi L\ meas(\Gamma_3)^{\frac{1}{2}}\Big), \quad \text{and} \quad k_2 = M_\psi L\ meas(\Gamma_3)^{\frac{1}{2}},$$

then Λ is an operator of $\mathcal{K}_1 \times \mathcal{K}_2$ into itself, and note that $\mathcal{K}_1 \times \mathcal{K}_2$ is a nonempty, convex and closed subset of $L^2(\Gamma_3) \times L^2(\Gamma_3)$. Since $L^2(\Gamma_3) \times L^2(\Gamma_3)$ is a reflexive space, $\mathcal{K}_1 \times \mathcal{K}_2$ is weakly compact. Using continuity of operators R and ϕ_L, (3.13) and (3.14), lemma 4.3, we deduce that Λ is weakly continuous. Hence, by Schauder's fixed point theorem the operator Λ has at least one fixed point. □

Proof of Theorem 4.1.

1) Existence. Let η^* be the fixed point of operator Λ. We denote by (u^*, φ^*) the solution of the variational problem PV_η for $\eta=\eta^*$. Using (4.13) and (4.18), it is easy to see that (u^*, φ^*) is a solution of PV. This proves the existence part of Theorem 4.1.

2) Uniqueness. We show next that, if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$, the solution is unique.

Let $x_1 = (u_1, \varphi_1)$, $x_2 = (u_2, \varphi_2) \in U$ the solution of problem (4.7), we have

$$(Ax_1, y - x_1)_X + \widetilde{j}(x_1, y) - \widetilde{j}(x_1, x_1) + \widetilde{\ell}(x_1, y - x_1) \ge (f_3, y - x_1)_X, \tag{4.27a}$$

$$(Ax_2, y - x_2)_X + \widetilde{j}(x_2, y) - \widetilde{j}(x_2, x_2) + \widetilde{\ell}(x_2, y - x_2) \ge (f_3, y - x_2)_X. \tag{4.27b}$$

We take $y = x_2$ in the first inequality, $y = x_1$ in the second, and add the two inequality to obtain

$$(A_1x_1 - Ax_2, x_1 - x_2)_X \le J + G, \tag{4.28}$$

where

$$J = \widetilde{j}(x_1, x_2) - \widetilde{j}(x_1, x_1) + \widetilde{j}(x_2, x_1) - \widetilde{j}(x_2, x_2), \tag{4.29a}$$

$$G = \widetilde{\ell}(x_1, x_2 - x_1) + \widetilde{\ell}(x_2, x_1 - x_2). \tag{4.29b}$$

From (4.29a), (4.4) and (3.18b), we have

$$\begin{aligned} J =& \int_{\Gamma_3} \mu(\|u_{1\tau}\|)\Big(|R\sigma_\nu(u_1)| - |R\sigma_\nu(u_2)|\Big)\Big(\|u_{1\tau}\| - \|u_{2\tau}\|\Big)da \\ &+ \int_{\Gamma_3} |R\sigma_\nu(u_2)|\Big(\mu(\|u_{1\tau}\|) - \mu(\|u_{\epsilon 2\tau}\|)\Big)\Big(\|u_{1\tau}\| - \|u_{2\tau}\|\Big)da, \end{aligned}$$

using (3.14), the continuity of R, (3.5) and (4.1), after some algebra, we obtain

$$J \leq \Big(\mu^* c_* c_0^2 + \|R\sigma_\nu(u_{\eta_1})\|_{L^\infty(\Gamma_3)} L_\mu c_0^2\Big)\|x_1 - x_2\|_X^2, \tag{4.30}$$

from (4.29b) and (4.5), we find

$$\begin{aligned} G =& \int_{\Gamma_3} \psi(u_{2\nu})\Big(\phi_L(\varphi_2 - \varphi_0) - \phi_L(\varphi_1 - \varphi_0)\Big)\Big(\varphi_1 - \varphi_2\Big)da \\ &+ \int_{\Gamma_3} \phi_L(\varphi_2 - \varphi_0)\Big(\psi(u_{2\nu}) - \psi(u_{1\nu})\Big)\Big(\varphi_1 - \varphi_2\Big)da, \end{aligned}$$

thus by using (3.13), the bounds $|\phi_L(\varphi_2 - \varphi_0)| \leq L$, the Lipschitz continuity of the function ϕ_L, (3.5), (3.7) and (4.1), we deduce

$$G \leq (M_\psi c_1^2 + L L_\psi c_0 c_1)\|x_1 - x_2\|_X^2. \tag{4.31}$$

Using (4.28), (4.30)-(4.31) and (4.15), hence there exists a constant $c_6 > 0$, such that

$$\|x_1 - x_2\|_X^2 \leq c_6(L_\mu + \mu^* + L_\psi L + M_\psi)\|x_1 - x_2\|_X^2.$$

Let $L^* = 1/c_6$, then if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$, therefore $x_1 = x_2$. □

5 Numerical approximation

In this section, we introduce and study the finite element approximation of the variational problem PV. Assume Ω is a polygonal domain, let τ^h be a regular family of triangular finite element partitions of $\overline{\Omega}$ that are compatible with the partition of the boundary decompositions $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and $\Gamma = \Gamma_a \cup \Gamma_b \cup \Gamma_3$, that is, any point when the boundary condition type changes is a vertex of the partitions, then the side lies entirely in $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, and $\overline{\Gamma}_a \cup \overline{\Gamma}_b \cup \overline{\Gamma}_3$. Corresponding to each partition τ^h. We denote by $\mathbb{P}_1(\Omega^e)$ the space of polynomials of global degree less or equal to one in Ω^e. Let us consider two finite-dimensional spaces $V^h \subset V$ and $W^h \subset W$, approximating the spaces V and W, respectively, that is

$$V^h = \{v^h \in C(\overline{\Omega})^d,\ v^h_{/\Omega^e} \in \mathbb{P}_1(\Omega^e)^d,\ \Omega^e \in \tau^h,\ v^h = 0,\ \text{on } \overline{\Gamma}_1\},$$
$$W^h = \{\psi^h \in C(\overline{\Omega}),\ \psi^h_{/\Omega^e} \in \mathbb{P}_1(\Omega^e),\ \ \Omega^e \in \tau^h,\ \psi^h = 0,\ \text{on } \overline{\Gamma}_a\}.$$

Here $h>0$ is a discretization parameter. Moreover, let us consider the nonempty, finite-dimensional, closed convex sets of admissible displacements with V^h, defined by $K^h = K \cap V^h$, i.e.,

$$K^h = \{v^h \in V^h,\ v^h_\nu \le 0,\ \text{on } \overline{\Gamma}_3\}.$$

In this section, c denotes a positive constant which depends on the problem data, but is independent of the discretization parameters h. We consider the following discrete approximation of problem PV:

Problem PV$^{\rm h}$: *Find a discrete displacement field $u^h \in K^h$, and a discrete electric potential $\varphi^h \in W^h$,such that*

$$\begin{aligned}
&\big(\mathfrak{F}\varepsilon(u^h), \varepsilon(v^h) - \varepsilon(u^h)\big)_{\mathcal{H}} + \big(\mathcal{E}^*\nabla\varphi^h, \varepsilon(v^h) - \varepsilon(u^h)\big)_{L^2(\Omega)^d} \\
&\quad + j(u^h, v^h) - j(u^h, u^h) \ge (f, v^h - u^h)_V, \quad \forall v^h \in K^h, &\text{(5.1a)}\\
&(\beta\nabla\varphi^h, \nabla\xi^h)_{L^2(\Omega)^d} - \big(\mathcal{E}\varepsilon(u^h), \nabla\xi^h\big)_{L^2(\Omega)^d} \\
&\quad + \ell(u^h, \varphi^h, \xi^h) = (q, \xi^h)_W, \qquad\qquad \forall \xi^h \in W^h. &\text{(5.1b)}
\end{aligned}$$

Using the assumptions of Theorem 4.1, it can be shown that Problem PV$^{\rm h}$ has a unique solution $(u^h, \varphi^h) \in K^h \times W^h$. Our interest lies in estimating the numerical errors. We first derive a Céa's type inequality (see [1,9]).

Theorem 5.1. *Let us denote by (u, φ) and (u^h, φ^h), the respective solutions to problem PV and PV^h. Under the assumptions of Theorem 4.1 with the same value of L^*, the following error estimates are obtained, for all $v^h \in K^h$, and $\psi^h \in W^h$,*

$$\begin{aligned}
&\|u - u^h\|_V + \|\varphi - \varphi^h\|_W \\
\le& c \inf_{(v^h,\xi^h)\in K^h\times W^h} \Big\{ \|u - v^h\|_V + \|\varphi - \xi^h\|_W + \|u - v^h\|_{L^2(\Gamma_3)^d} \\
&+ \|\varphi - \xi^h\|_{L^2(\Gamma_3)} + \Big(\|\mathfrak{F}\varepsilon(u)\|^{\frac{1}{2}}_{\mathcal{H}} + \|\mathcal{E}^*\nabla\varphi^h\|^{\frac{1}{2}}_{\mathcal{H}} + \|f\|^{\frac{1}{2}}_V\Big)\|u - v^h\|^{\frac{1}{2}}_V \\
&+ \Big(\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\Big)^{\frac{1}{2}}\|u - v^h\|^{\frac{1}{2}}_{L^2(\Gamma_3)^d}\Big\}. &\text{(5.2)}
\end{aligned}$$

Proof. Taking $\xi = \xi^h \in W^h$ in (3.20b), and subtracting it to (5.1b), we obtain that

$$\Big(\beta\nabla(\varphi - \varphi^h), \nabla\xi^h\Big)_{L^2(\Omega)^d} - \Big(\mathcal{E}\varepsilon(u - u^h), \nabla\xi^h\Big)_{L^2(\Omega)^d} + \ell(u, \varphi, \xi^h) - \ell(u^h, \varphi^h, \xi^h) = 0,$$

thus, we have

$$\begin{aligned}
&\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\xi^h-\varphi)\Big)_{L^2(\Omega)^d}+\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\varphi-\varphi^h)\Big)_{L^2(\Omega)^d}\\
&-\Big(\mathcal{E}\varepsilon(u-u^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d}-\Big(\mathcal{E}\varepsilon(u-u^h),\nabla(\varphi-\varphi^h)\Big)_{L^2(\Omega)^d}\\
&+\ell(u,\varphi,\xi^h-\varphi)+\ell(u,\varphi,\varphi-\varphi^h)-\ell(u^h,\varphi^h,\xi^h-\varphi)\\
&-\ell(u^h,\varphi^h,\varphi-\varphi^h)=0,\qquad\forall\xi^h\in W^h,
\end{aligned}$$

and therefore for all $\xi^h \in W^h$, it follows that

$$\begin{aligned}
&\Big(\mathcal{E}\varepsilon(u-u^h),\nabla(\varphi-\varphi^h)\Big)_{L^2(\Omega)^d}\\
=&\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\varphi-\varphi^h)\Big)_{L^2(\Omega)^d}-\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d}\\
&+\Big(\mathcal{E}\varepsilon(u-u^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d}+\ell(u,\varphi,\varphi-\varphi^h)-\ell(u^h,\varphi^h,\varphi-\varphi^h)\\
&+\ell(u,\varphi,\xi^h-\varphi)-\ell(u^h,\varphi^h,\xi^h-\varphi).
\end{aligned}\tag{5.3}$$

Next, choosing $v=u^h\in K^h$ in (3.20a), we find

$$\begin{aligned}
&\Big(\mathfrak{F}\varepsilon(u),\varepsilon(u)-\varepsilon(u^h)\Big)_{\mathcal{H}}+\Big(\mathcal{E}^*\nabla\varphi,\varepsilon(u)-\varepsilon(u^h)\Big)_{L^2(\Omega)^d}\\
\le&j(u,u^h)-j(u,u)+(f,u-u^h)_V.
\end{aligned}\tag{5.4}$$

We rewrite now to estimate variational inequality (5.1a) as follows :

$$\begin{aligned}
&\Big(-\mathfrak{F}\varepsilon(u^h),\varepsilon(u-u^h)\Big)_{\mathcal{H}}+\Big(-\mathcal{E}^*\nabla\varphi^h,\varepsilon(u-u^h)\Big)_{L^2(\Omega)^d}\\
\le&\Big(\mathfrak{F}\varepsilon(u^h),\varepsilon(v^h-u)\Big)_{\mathcal{H}}+\Big(\mathcal{E}^*\nabla\varphi^h,\varepsilon(v^h-u)\Big)_{L^2(\Omega)^d}\\
&+j(u^h,v^h)-j(u^h,u^h)+(f,u^h-v^h)_V,\qquad\forall v^h\in K^h.
\end{aligned}\tag{5.5}$$

Adding (5.4) and (5.5), we obtain

$$\begin{aligned}
&\Big(\mathfrak{F}\varepsilon(u)-\mathfrak{F}\varepsilon(u^h),\varepsilon(u-u^h)\Big)_{\mathcal{H}}+\Big(\nabla(\varphi-\varphi^h),\mathcal{E}\varepsilon(u-u^h)\Big)_{L^2(\Omega)^d}\\
\le&\Big(\mathfrak{F}\varepsilon(u)-\mathfrak{F}\varepsilon(u^h),\varepsilon(u-v^h)\Big)_{\mathcal{H}}+\Big(\mathfrak{F}\varepsilon(u),\varepsilon(v^h-u)\Big)_{\mathcal{H}}\\
&+\Big(\mathcal{E}^*\nabla\varphi^h,\varepsilon(v^h-u)\Big)_{L^2(\Omega)^d}+j(u,u^h)-j(u,u)+j(u^h,v^h)\\
&-j(u^h,u^h)-(f,v^h-u)_V,\qquad\forall v^h\in K^h.
\end{aligned}$$

Keeping in mind (5.3), we deduce

$$\begin{aligned}
&\Big(\mathfrak{F}\varepsilon(u)-\mathfrak{F}\varepsilon(u^h),\varepsilon(u-u^h)\Big)_{\mathcal{H}}+\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\varphi-\varphi^h)\Big)_{L^2(\Omega)^d}\\
\le&\Big(\mathfrak{F}\varepsilon(u)-\mathfrak{F}\varepsilon(u^h),\varepsilon(u-v^h)\Big)_{\mathcal{H}}+\Big(\mathfrak{F}\varepsilon(u),\varepsilon(v^h-u)\Big)_{\mathcal{H}}
\end{aligned}$$

$$\begin{aligned}&+\Big(\mathcal{E}^*\nabla\varphi^h,\varepsilon(v^h-u)\Big)_{L^2(\Omega)^d}+\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d}\\&-\Big(\mathcal{E}\varepsilon(u-u^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d}+j(u,u^h)-j(u,u)+j(u^h,v^h)\\&-j(u^h,u^h)-(f,v^h-u)_V-\ell(u,\varphi,\varphi-\varphi^h)+\ell(u^h,\varphi^h,\varphi-\varphi^h)\\&-\ell(u,\varphi,\xi^h-\varphi)+\ell(u^h,\varphi^h,\xi^h-\varphi),\qquad\forall v^h\in K^h,\ \forall\xi^h\in W^h.\end{aligned}$$

Using the assumptions (3.10)-(3.12), and the previous inequality, it follows that,

$$m_{\mathfrak{F}}\|u-u^h\|_V^2+m_\beta\|\varphi-\varphi^h\|_W^2\le R_1+R_2+R_3+R_4+R_5, \tag{5.6}$$

where

$$\begin{aligned}R_1=&\Big(\mathfrak{F}\varepsilon(u)-\mathfrak{F}\varepsilon(u^h),\varepsilon(u-v^h)\Big)_{\mathcal{H}}+\Big(\beta\nabla(\varphi-\varphi^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d}\\&-\Big(\mathcal{E}\varepsilon(u-u^h),\nabla(\varphi-\xi^h)\Big)_{L^2(\Omega)^d},\\R_2=&\Big(\mathfrak{F}\varepsilon(u),\varepsilon(v^h-u)\Big)_{\mathcal{H}}+\Big(\mathcal{E}^*\nabla\varphi^h,\varepsilon(v^h-u)\Big)_{L^2(\Omega)^d}+j(u,v^h)\\&-j(u,u)-(f,v^h-u)_V,\\R_3=&\,j(u,u^h)-j(u^h,u^h)+j(u^h,u)-j(u,u),\\R_4=&\,j(u^h,v^h)-j(u,v^h)+j(u,u)-j(u^h,u),\\R_5=&\,\ell(u^h,\varphi^h,\varphi-\varphi^h)-\ell(u,\varphi,\varphi-\varphi^h)+\ell(u^h,\varphi^h,\xi^h-\varphi)-\ell(u,\varphi,\xi^h-\varphi).\end{aligned}$$

Let us estimate each of the five terms. For the first term, and by using the properties of the operators $\mathfrak{F}$, β and $\mathcal{E}$, we have

$$\begin{aligned}|R_1|\le c\big\{&\|\varepsilon(u-u^h)\|_{\mathcal{H}}\|\varepsilon(u-v^h)\|_{\mathcal{H}}+\|\varphi-\varphi^h\|_W\|\varphi-\xi^h\|_W\\&+\|\varepsilon(u-u^h)\|_{\mathcal{H}}\|\varphi-\xi^h\|_W\big\},\end{aligned} \tag{5.7}$$

from the property (3.14), it follows

$$\begin{aligned}|R_2|\le&\|\mathfrak{F}\varepsilon(u)\|_{\mathcal{H}}\|\varepsilon(u-v^h)\|_{\mathcal{H}}+\|\mathcal{E}^*\nabla\varphi^h\|_{\mathcal{H}}\|\varepsilon(u-v^h)\|_{\mathcal{H}}\\&+\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\|u-v^h\|_{L^2(\Gamma_3)^d}\\&+\|f\|_V\|u-v^h\|_V.\end{aligned} \tag{5.8}$$

Since

$$\begin{aligned}R_3=&\int_{\Gamma_3}(\mu(\|u_\tau\|)|R\sigma_\nu(u)|-\mu(\|u_\tau^h\|)|R\sigma_\nu(u^h)|)\,(\|u_\tau^h\|-\|u_\tau\|)d\gamma\\=&\int_{\Gamma_3}|R\sigma_\nu(u)|(\mu(\|u_\tau\|)-\mu(\|u_\tau^h\|))\,(\|u_\tau^h\|-\|u_\tau\|)d\gamma\\&+\int_{\Gamma_3}\mu(\|u_\tau^h\|)(|R\sigma_\nu(u)|-|R\sigma_\nu(u^h)|)\,(\|u_\tau^h\|-\|u_\tau\|)d\gamma,\end{aligned}$$

and using (3.14), we have

$$|R_3|\le c(\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}L_\mu+\mu^*)\|u-u^h\|_V^2. \tag{5.9}$$

Similarly

$$\begin{aligned} R_4 =& \int_{\Gamma_3} \Big(\mu(\|u^h_\tau\|)|R\sigma_\nu(u^h)| - \mu(\|u_\tau\|)|R\sigma_\nu(u)|\Big)(\|v^h_\tau\| - \|u_\tau\|)d\gamma \\ =& \int_{\Gamma_3} |R\sigma_\nu(u)|\Big(\mu(\|u^h_\tau\|) - \mu(\|u_\tau\|)\Big)(\|v^h_\tau\| - \|u_\tau\|)d\gamma \\ &+ \int_{\Gamma_3} \mu(\|u^h_\tau\|)\Big(|R\sigma_\nu(u^h)| - |R\sigma_\nu(u)|\Big)(\|v^h_\tau\| - \|u_\tau\|)d\gamma, \end{aligned}$$

and then

$$|R_4| \le c\|u - u^h\|_V \|u - v^h\|_{L^2(\Gamma_3)^d}, \tag{5.10}$$

from (3.18a), we find

$$\begin{aligned} R_5 =& -\int_{\Gamma_3} (\psi(u_\nu)\phi_L(\varphi - \varphi_0) - \psi(u^h_\nu)\phi_L(\varphi^h - \varphi_0))(\varphi - \varphi^h)d\gamma \\ &- \int_{\Gamma_3} (\psi(u_\nu)\phi_L(\varphi - \varphi_0) - \psi(u^h_\nu)\phi_L(\varphi^h - \varphi_0))(\xi^h - \varphi)d\gamma \\ =& -\int_{\Gamma_3} \psi(u_\nu)(\phi_L(\varphi - \varphi_0) - \phi_L(\varphi^h - \varphi_0))(\varphi - \varphi^h)d\gamma \\ &- \int_{\Gamma_3} \phi_L(\varphi^h - \varphi_0)(\psi(u_\nu) - \psi(u^h_\nu))(\varphi - \varphi^h)d\gamma \\ &- \int_{\Gamma_3} \psi(u_\nu)(\phi_L(\varphi - \varphi_0) - \phi_L(\varphi^h - \varphi_0))(\xi^h - \varphi)d\gamma \\ &- \int_{\Gamma_3} \phi_L(\varphi^h - \varphi_0)(\psi(u_\nu) - \psi(u^h_\nu))(\xi^h - \varphi)d\gamma, \end{aligned}$$

thus by using (3.13), the bounds $|\phi_L(\varphi - \varphi_0)| \le L$, and the Lipschitz continuity of the function ϕ_L, we obtain

$$\begin{aligned} |R_5| \le& M_\psi \int_{\Gamma_3} |\varphi - \varphi^h|^2 d\gamma + LL_\psi \int_{\Gamma_3} |\varphi - \varphi^h||u_\nu - u^h_\nu|d\gamma \\ &+ M_\psi \int_{\Gamma_3} |\varphi - \varphi^h||\xi^h - \varphi|d\gamma + LL_\psi \int_{\Gamma_3} |u_\nu - u^h_\nu||\xi^h - \varphi|d\gamma, \end{aligned}$$

therefore

$$\begin{aligned} |R_5| \le& M_\psi c_1 \|\varphi - \varphi^h\|^2_W + LL_\psi c_0 c_1 \|u - u^h\|_V \|\varphi - \varphi^h\|_W \\ &+ M_\psi c_1 \|\varphi - \varphi^h\|_W \|\varphi - \xi^h\|_{L^2(\Gamma_3)} + LL_\psi c_0 \|u - u^h\|_V \|\varphi - \xi^h\|_{L^2(\Gamma_3)}. \end{aligned} \tag{5.11}$$

Applying now the η-inequality

$$ab \le \eta a^2 + \frac{1}{4\eta} b^2,$$

and using the bounds (5.7)-(5.11), after some calculations, it follows that

$$\begin{aligned}
&\|u-u^h\|_V^2+\|\varphi-\varphi^h\|_W^2\\
\le& c\Big\{\|u-v^h\|_V^2+\|\varphi-\xi^h\|_W^2+\|u-v^h\|_{L^2(\Gamma_3)^d}^2+\|\varphi-\xi^h\|_{L^2(\Gamma_3)}^2\\
&+\big(\|\mathfrak{F}\varepsilon(u)\|_{\mathcal{H}}+\|\mathcal{E}^*\nabla\varphi^h\|_{\mathcal{H}}+\|f\|_V\big)\|u-v^h\|_V\\
&+\big(\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\big)\|u-v^h\|_{L^2(\Gamma_3)^d}\Big\},
\end{aligned}$$

so the inequality (5.2) holds. □

The inequality (5.2) is a basis for deriving error estimation and convergence analysis. In an analogous way, we can improve the estimate (5.2) under the regularity assumption $\sigma_\tau \in L^2(\Gamma_3)^d$. In this case, integrating by parts the Eq. (2.3) and using the constitutive law (2.1), and the boundary conditions (2.5)-(2.8), we obtain

$$R_2=\int_{\Gamma_3}\sigma_\tau(v_\tau^h-u_\tau)+\big(\mathcal{E}^*\nabla(\varphi^h-\varphi),\varepsilon(v^h-u)\big)_{L^2(\Omega)^d}+j(u,v^h)-j(u,u),$$

thus using Cauchy inequality given above, we have

$$\begin{aligned}
|R_2|\le&\ \eta\|\varphi-\varphi^h\|_W^2+\frac{1}{4\eta}\|\varepsilon(u-v^h)\|_{\mathcal{H}}^2\\
&+\big(\|\sigma_\tau\|_{L^2(\Gamma_3)^d}+\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\big)\|u-v^h\|_{L^2(\Gamma_3)^d}, \qquad (5.12)
\end{aligned}$$

to replace (5.12). As a result we have the following variant of Theorem 5.1.

Theorem 5.2. *Under the assumptions of Theorem 4.1 with the same value of L^*, assume additionally $\sigma_\tau\in L^2(\Gamma_3)^d$. Then for some constant $c>0$, we have*

$$\begin{aligned}
&\|u-u^h\|_V+\|\varphi-\varphi^h\|_W\\
\le& c\inf_{(v^h,\xi^h)\in K^h\times W^h}\Big\{\|u-v^h\|_V+\|\varphi-\xi^h\|_W+\|u-v^h\|_{L^2(\Gamma_3)^d}+\|\varphi-\xi^h\|_{L^2(\Gamma_3)}\\
&+\big(\|\sigma_\tau\|_{L^2(\Gamma_3)^d}+\|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)}\|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\big)^{\frac12}\|u-v^h\|_{L^2(\Gamma_3)^d}^{\frac12}\Big\}. \qquad (5.13)
\end{aligned}$$

To estimate the errors provided by the approximation of the finite element spaces V^h and W^h, we need to make an additional assumption on the regularity of the solution

$$u\in H^2(\Omega)^d,\quad u_{|\Gamma_3}\in H^2(\Gamma_3)^d,\quad \varphi\in H^2(\Omega),\quad \varphi_{|\Gamma_3}\in H^2(\Gamma_3). \qquad (5.14)$$

Again denoting $\Pi^h u$ and $\Pi^h\varphi$ the standard finite element interpolation operators of u and φ, respectively, then we have the interpolation error estimate (cf. [7])

$$\begin{aligned}
\|u-\Pi^h u\|_V&\le ch|u|_{H^2(\Omega)^d},\\
\|\varphi-\Pi^h\varphi\|_W&\le ch|\varphi|_{H^2(\Omega)}.
\end{aligned}$$

The restriction of the partitions τ^h on $\overline{\Gamma}_3$ induces a regular family of finite-element partitions of $\overline{\Gamma}_3$. So we also have the interpolation error estimate

$$\|u - \Pi^h u\|_{L^2(\Gamma_3)^d} \leq ch^2 |u|_{H^2(\Gamma_3)^d},$$
$$\|\varphi - \Pi^h \varphi\|_{L^2(\Gamma_3)} \leq ch^2 |\varphi|_{H^2(\Gamma_3)}.$$

We notice that for $v \in K \cap C(\overline{\Omega})^d$, $\Pi^h v \in K^h$.

Therefor, under the regularity assumption (5.14), we have the following error estimate:

$$\begin{aligned} &\|u - u^h\|_V + \|\varphi - \varphi^h\|_W \\ \leq & ch\Big\{ |u|_{H^2(\Omega)^d} + |\varphi|_{H^2(\Omega)} + h|u|_{H^2(\Gamma_3)^d} + h\,|\varphi|_{H^2(\Gamma_3)} \\ &+ \big(\|\sigma_\tau\|_{L^2(\Gamma_3)^d} + \|R\sigma_\nu(u)\|_{L^\infty(\Gamma_3)} \|\mu(\|u_\tau\|)\|_{L^2(\Gamma_3)}\big)^{\frac{1}{2}} |u|^{\frac{1}{2}}_{H^2(\Gamma_3)^d} \Big\}. \end{aligned}$$

The finite element system (5.1a)-(5.1b) can be approximated by a fixed point iteration method. This follows from a discrete analogue of the proof of Theorem 4.1. Given an initial guess $(u_0^h,\ \varphi_0^h)$, we define a sequence $(u_n^h,\ \varphi_n^h) \in K^h \times W^h$, for all $n \in \mathbb{N}$ recursively by

$$\begin{aligned} &\Big(\mathfrak{F}\varepsilon(u^h_{n+1}), \varepsilon(v^h) - \varepsilon(u^h_{n+1})\Big)_{\mathcal{H}} + \Big(\mathcal{E}^* \nabla \varphi^h_{n+1}, \varepsilon(v^h) - \varepsilon(u^h_{n+1})\Big)_{L^2(\Omega)^d} \\ &\quad + j(u^h_n, v^h) - j(u^h_n, u^h_{n+1}) \geq (f, v^h - u^h_{n+1})_V, \quad \forall v^h \in K^h, \end{aligned} \tag{5.15a}$$

$$\begin{aligned} &(\beta \nabla \varphi^h_{n+1}, \nabla \xi^h)_{L^2(\Omega)^d} - \Big(\mathcal{E}\varepsilon(u^h_{n+1}), \nabla \xi^h\Big)_{L^2(\Omega)^d} \\ &\quad + \ell(u^h_n, \varphi^h_n, \xi^h) = (q, \xi^h)_W, \qquad \forall \xi^h \in W^h. \end{aligned} \tag{5.15b}$$

We have the following convergence result.

Theorem 5.3. *Under the assumptions of Theorem* (4.1) *with the same value of* L^*, *the iteration method* (5.15a)-(5.15b) *converges :*

$$\|u^h_n - u^h\|_V \to 0, \quad as \quad n \to \infty,$$
$$\|\varphi^h_n - \varphi^h\|_W \to 0, \quad as \quad n \to \infty.$$

Furthermore, for some constant $0 < k < 1$, *we have the estimate*

$$\begin{aligned} \|u^h_n - u^h\|_V \leq ck^n, \\ \|\varphi^h_n - \varphi^h\|_W \leq ck^n. \end{aligned} \tag{5.16}$$

Proof. Using lemma 4.1, it is easy to see that
(i) The couple $x^h = (u^h, \varphi^h)$ is a solution to problem $\mathrm{PV^h}$, if and only if :

$$\begin{aligned} &(Ax^h, y^h - x^h)_X + \widetilde{j}(x^h, y^h) - \widetilde{j}(x^h, x^h) + \widetilde{\ell}(x^h, y - x^h) \\ \geq &(f, y^h - x^h)_X, \qquad \forall y^h = (v^h, \xi^h) \in K^h \times W^h. \end{aligned} \tag{5.17}$$

(ii) The couple $x_n^h = (u_n^h; \varphi_n^h)$ is a solution to problems (5.15a)-(5.15b) if and only if

$$\begin{aligned}&(Ax_{n+1}^h, y^h - x_{n+1}^h)_X + \widetilde{j}(x_n^h, y^h) - \widetilde{j}(x_n^h, x_{n+1}^h) + \widetilde{\ell}(x_n^h, y^h - x_{n+1})\\ \geq&(f, y - x_{n+1}^h)_X, \qquad \forall y = (v^h, \xi^h) \in K^h \times W^h. \end{aligned} \tag{5.18}$$

We take $y^h = x_{n+1}^h$ in (5.17), $y^h = x^h$ in (5.18), and adding, we have

$$\begin{aligned}&(Ax^h - Ax_{n+1}^h, x^h - x_{n+1}^h)_X\\ \leq&\widetilde{j}(x^h, x_{n+1}^h) - \widetilde{j}(x^h, x^h) + \widetilde{j}(x_n^h, x^h) - \widetilde{j}(x_n^h, x_{n+1}^h)\\ &+ \widetilde{\ell}(x_n^h, x^h - x_{n+1}^h)_X - \widetilde{\ell}(x^h, x^h - x_{n+1}^h)_X.\end{aligned}$$

Then as in the proof of the uniqueness of Theorem 4.1, we can derive the estimate

$$\|x^h - x_{n+1}^h\|_X \leq c_6(L_\mu + \mu^* + L_\psi L + M_\psi)\|x^h - x_n^h\|_X,$$

thus

$$\|x^h - x_{n+1}^h\|_X \leq \frac{(L_\mu + \mu^* + L_\psi L + M_\psi)}{L^*}\|x^h - x_n^h\|_X.$$

Under the stated assumption, $k \equiv (L_\mu + \mu^* + L_\psi L + M_\psi)/L^* < 1$, and we have the estimate (5.16). □

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