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Adaptive hp-FEM with Arbitrary-Level Hanging Nodes for Maxwell's Equations

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Abstract. Adaptive higher-order finite element methods (hp-FEM) are well known for their potential of exceptionally fast (exponential) convergence. However, most hp-FEM codes remain in an academic setting due to an extreme algorithmic complexity of hp-adaptivity algorithms. This paper aims at simplifying hp-adaptivity for $\boldsymbol{H}$(curl)-conforming approximations by presenting a novel technique of arbitrary-level hanging nodes. The technique is described and it is demonstrated numerically that it makes adaptive hp-FEM more efficient compared to hp-FEM on regular meshes and meshes with one-level hanging nodes.

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Key words: hp-FEM, arbitrary-level hanging nodes, irregular meshes, higher-order edge elements, Maxwell's equations.

1 Introduction

Nowadays, vector-valued finite elements with continuous tangential components on element interfaces (edge elements) are a standard tool for the solution of Maxwell's equations in various cavity devices such as waveguides, resonators, microwave ovens, and other models. Edge elements are based on differential forms introduced in late 1950s by H. Whitney [19], in the context of differential geometry. Apparently, the first link between the Whitney forms and computational electromagnetics was made in 1984 by P. R. Kotiuga in his thesis [9]. A nice monograph on this subject is [3].

Adaptive higher-order finite element methods (hp-FEM) based on higher-order edge elements belong to the youngest topics in computational electromagnetics (see,

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e.g., [9, 10, 16] and the references therein). Especially for problems involving important small-scale phenomena such as singularities or steep gradients along internal or boundary layers, the efficiency gap between adaptive hp-FEM and standard adaptive low-order FEM can be impressive. On the other hand, these methods are not used widely by practitioners yet due to their high algorithmic complexity. From this point of view, the design of simple hp-adaptivity algorithms is of crucial importance.

It is worth mentioning that hp-adaptivity is profoundly different from h-adaptivity due to a large number of element refinement options per element (around 100 in 2D and several hundred in 3D). This number depends on multiple factors such as whether one allows anisotropic refinements in space and anisotropic (directionally different) polynomial degrees in quadrilateral/hexahedral elements, how much the polynomial degree is allowed to vary in subelements after an element is refined in space, etc. Standard a-posteriori error estimates used for h-adaptivity, that only provide an information about the magnitude of error in elements, do not help to select an optimal element refinement in hp-adaptivity. For that, one needs a much better information about the error, namely *its shape in every element*. In principle, this information might be reconstructed from suitable a-posteriori estimates of higher derivatives of the solution, but this would be extremely difficult and the authors are not aware of any such work. Currently, the two major approaches to guiding adaptivity in higher-order finite element methods are:

1. *Computing a reference solution* on a globally refined mesh [11, 14]. This approach is computationally expensive but on the other hand it works for *any equation including multiphysics coupled problems where no standard a-posteriori error estimates are available* [7, 15, 17, 18].
2. *Estimating analyticity* of the solution in every element in order to decide whether an h- and p-refinement should be done [8]. This technique requires additional equation-dependent tuning parameters, and it does not allow variable polynomial degrees in subelements when an element is refined in space.

In this paper we use the former approach, and extend a novel technique of arbitrary-level hanging nodes [13] from standard H^1-conforming (continuous scalar) approximations to vector-valued approximations in $\boldsymbol{H}(\text{curl})$. This technique is a valuable addition to existing adaptivity algorithms since it makes it possible to refine any element in the mesh locally, without affecting its neighbors. In turn one can design simple hp-adaptivity algorithms that work in an element-by-element fashion. In other words, when refining an element, one never has to refine neighboring mesh elements to keep the mesh regular. Note that this is impossible with algorithms employing regular meshes such as [4] or meshes containing one-level hanging nodes [6], since in these cases one has to deal with unwanted, regularity-enforced additional refinements.

There exist several implementations of the technique of multiple-level hanging nodes for second-order elliptic problems [6, 12, 13], but to our best knowledge, the technique [13] is the only one to work independently of the underlying higher-order shape functions.