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Analysis and Invariant Properties of a Lattice Boltzmann Method

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Abstract. We investigate a two-relaxation-time (TRT) lattice Boltzmann algorithm with the asymptotic analysis technique. The results are used to analyze invariance properties of the method. In particular, we focus on time dependent Stokes and Navier-Stokes problems.

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1 Introduction

The investigations reported here are motivated by the seminal paper [6,8] on the two-relaxation-time (TRT) lattice Boltzmann algorithm. Compared to the full MRT collision operator [4,5,11,17], the TRT approach has the advantage of being much closer in spirit to a BGK operator which alleviates the implementation and improves the efficiency of the algorithm. On the other hand, it outperforms the BGK [2] method because the extra relaxation parameter which is not required for consistency can be used to improve the stability, to reduce certain error terms or to achieve specific invariances.

The latter case has been carefully studied in [6] where the TRT algorithm is used to approximate the stationary Stokes or Navier-Stokes equation and where certain invariances of the algorithm are explained in detail. To see more clearly, which aspects of the algorithm lead to a loss of these invariance in case of the instationary equations is one goal of the present work.

We approach this goal by deriving the equation for the leading order error of the TRT algorithm using the asymptotic analysis method [12,14,19]. One advantage of this approach is the very transparent explanation of the relation between the lattice Boltzmann output and the solution of the incompressible Stokes or Navier-Stokes equation

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which can even be used for a convergence proof [15, 16]. Secondly, the fact that the asymptotic analysis is based on a regular expansion with coefficients that are fully expanded (as opposed to the Chapman Enskog approach [1,5,7]) allows to easily derive necessary conditions for invariances of the underlying algorithm.

Compared to earlier works, we present a simplification of the asymptotic analysis which is also related to invariance properties. In fact, whenever the solution of a singularly perturbed equation satisfies an additional, dependent relation which becomes independent in the limit, the asymptotic analysis can be simplified. Since the lattice Boltzmann algorithm can be viewed as a discretization of a singularly perturbed finite velocity Boltzmann equation with mass and momentum balance as additional relation, this idea can be employed.

We conclude the introduction with a short outline of the article. In section 2 we introduce several examples of invariances which will be considered later in case of the lattice Boltzmann algorithm. Moreover, we address how asymptotic analysis can be used to derive necessary conditions for invariance and how invariance properties can help in the analysis of singularly perturbed problems. In section 3, we introduce a complete TRT lattice Boltzmann algorithm, carry out the consistency analysis and close with a summary of the post processing needed to extract the Stokes or Navier-Stokes fields form the lattice Boltzmann variables. In the final section 4, we use the equation of the leading order error to check the invariance properties of the TRT algorithm with a special focus on the case of instationary equations.

2 Invariant properties

This article deals with some specific aspects of the general problem how to approximate equations $E(U)$=0 which are accompanied by relevant additional conditions $A(U)$=0. We begin with some examples for this general framework, discuss numerical approximations on a general level and conclude with some comments on the analysis of lattice Boltzmann methods, which can benefit from using the availability of additional conditions.

2.1 Examples

2.1.1 The harmonic oscillator

A famous example for an equation with meaningful additional condition is the harmonic oscillator

$$\ddot{U} + U = 0, \qquad U(0) = 1, \qquad \dot{U}(0) = 0,$$

where the total energy is conserved, i.e.

$$(\dot{U})^2 + U^2 = 1.$$