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A Posteriori Error Estimate for Stabilized Low-Order Mixed FEM for the Stokes Equations

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Abstract. This paper is concerned with a stabilized approach to low-order mixed finite element methods for the Stokes equations. We will provide a posteriori error analysis for the method. We present two a posteriori error indicators which will be demonstrated to be globally upper and locally lower bounds for the error of the finite element discretization. Finally two numerical experiments will be carried out to show the efficiency on constructing adaptive meshes.

AMS subject classifications: 65N15, 65N30, 65N50
Key words: A posteriori error estimate, stabilized low-order mixed FEM, error indicator, Stokes equations.

1 Introduction

In engineering practice we often make use of the low-order mixed finite element methods because of their advantages in computation. However the discretization form of the Stokes equations with these elements usually does not satisfy the inf-sup condition. As a result many methods have been proposed to fix the deficiency, such as the penalty method [1] and pressure gradient method [2]. It is noted that [3] presents a new stabilized approach to the equations. After adding an stabilized term $G(p,q)$ to the variational formulation of the Stokes equations, the discretization form can satisfy the inf-sup condition and thus has a unique solution. Comparing with other methods it is much easier in computation because it does not need any approximation of derivatives, or mesh-dependent parameter.

It is practically important to make a posteriori analysis for a numerical method. As is known that the most important efficiency of a posteriori analysis lays on constructing adaptive meshes [4,5]. Babuška and Rheinboldt started the pioneering work

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about the posteriori error estimation for finite element methods for two point elliptic boundary value problem [6], see also their work near that period [7, 8]. After that, many works have been done in this area, see, e.g., [9, 10]. Verfürth derived a posteriori error analysis for the Stokes equations [11, 12] and Navier-Stokes equations [13]. In [11] he presented two a posteriori error estimators for the mini-element discretization of the Stokes equations and proved that they were upper bound and local lower bound of the finite element error. These indicators are often changed to be applied to other situations, see, e.g., [14]. In addition the a posteriori error analysis of many other discretization forms has been done, see, e.g., [15].

In this paper we present a posteriori error analysis of the stabilized method mentioned in the first paragraph. Our work is similar to a posteriori error analysis of a penalty method [14, 16]. However, one of the useful points of our method is that it is parameter-free. We give two a posteriori error estimators and show that they are equivalent to the errors.

The paper is organized as follows. In Section 2, we give a review of the stabilized method for low-order mixed finite element method. Here we choose the $P1$-$P1$ velocity-pressure pairs. In Section 3, we prove the equivalence between the a posteriori error estimators and the error of the finite element method. In Section 4, two numerical experiments show the efficiency of our analysis, mainly in constructing adaptive meshes.

2 The stabilized low-order mixed FEM for the Stokes equation

Let Ω be a bounded, connected, polygonal domain in $\mathbb{R}^2$. We consider the Stokes equations

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f}, \qquad \text{in } \Omega, \tag{2.1a}$$

$$\nabla\cdot\mathbf{u}=0, \qquad \text{in } \Omega, \tag{2.1b}$$

$$\mathbf{u}=\mathbf{0}, \qquad \text{on } \Gamma:=\partial\Omega. \tag{2.1c}$$

We use the standard notations $H^k(\Omega)$, $\|\cdot\|_k$, $(\cdot,\cdot)_k$, $k\geq 0$ denote the usual Soblev space, the standard Soblev norm and inner product, respectively. Especially when $k=0$, $L^2(\Omega)=H^0(\Omega)$ denotes the usual Lebegsgue space. We also introduce the spaces

$$H_0^1(\Omega)=\Big\{\varphi\in H^1(\Omega);\quad \varphi=0,\ \ \text{on } \Gamma\Big\},$$

$$L_0^2(\Omega)=\Big\{\varphi\in L^2(\Omega);\quad \int_\Omega\varphi=0\Big\}.$$

Next, we give the mixed variational form of (2.1). Find

$$(\mathbf{u},p)\in H_0^1(\Omega)^2\times L_0^2(\Omega),$$