Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **2**, No. 6, pp. 798-809

DOI: 10.4208/aamm.09-m0995
December 2010

A Posteriori Error Estimate for Stabilized Low-Order Mixed FEM for the Stokes Equations

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Received 21 October 2009; Accepted (in revised version) 14 May 2010
Available online 26 August 2010

Abstract. This paper is concerned with a stabilized approach to low-order mixed finite element methods for the Stokes equations. We will provide a posteriori error analysis for the method. We present two a posteriori error indicators which will be demonstrated to be globally upper and locally lower bounds for the error of the finite element discretization. Finally two numerical experiments will be carried out to show the efficiency on constructing adaptive meshes.

AMS subject classifications: 65N15, 65N30, 65N50
Key words: A posteriori error estimate, stabilized low-order mixed FEM, error indicator, Stokes equations.

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1 Introduction

In engineering practice we often make use of the low-order mixed finite element methods because of their advantages in computation. However the discretization form of the Stokes equations with these elements usually does not satisfy the inf-sup condition. As a result many methods have been proposed to fix the deficiency, such as the penalty method [1] and pressure gradient method [2]. It is noted that [3] presents a new stabilized approach to the equations. After adding an stabilized term $G(p,q)$ to the variational formulation of the Stokes equations, the discretization form can satisfy the inf-sup condition and thus has a unique solution. Comparing with other methods it is much easier in computation because it does not need any approximation of derivatives, or mesh-dependent parameter.

It is practically important to make a posteriori analysis for a numerical method. As is known that the most important efficiency of a posteriori analysis lays on constructing adaptive meshes [4,5]. Babuška and Rheinboldt started the pioneering work

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about the posteriori error estimation for finite element methods for two point elliptic boundary value problem [6], see also their work near that period [7,8]. After that, many works have been done in this area, see, e.g., [9,10]. Verfürth derived a posteriori error analysis for the Stokes equations [11,12] and Navier-Stokes equations [13]. In [11] he presented two a posteriori error estimators for the mini-element discretization of the Stokes equations and proved that they were upper bound and local lower bound of the finite element error. These indicators are often changed to be applied to other situations, see, e.g., [14]. In addition the a posteriori error analysis of many other discretization forms has been done, see, e.g., [15].

In this paper we present a posteriori error analysis of the stabilized method mentioned in the first paragraph. Our work is similar to a posteriori error analysis of a penalty method [14,16]. However, one of the useful points of our method is that it is parameter-free. We give two a posteriori error estimators and show that they are equivalent to the errors.

The paper is organized as follows. In Section 2, we give a review of the stabilized method for low-order mixed finite element method. Here we choose the $P1$-$P1$ velocity-pressure pairs. In Section 3, we prove the equivalence between the a posteriori error estimators and the error of the finite element method. In Section 4, two numerical experiments show the efficiency of our analysis, mainly in constructing adaptive meshes.

2 The stabilized low-order mixed FEM for the Stokes equation

Let Ω be a bounded, connected, polygonal domain in $\mathbb{R}^2$. We consider the Stokes equations

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f}, \qquad \text{in } \Omega, \tag{2.1a}$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{in } \Omega, \tag{2.1b}$$

$$\mathbf{u} = \mathbf{0}, \qquad \text{on } \Gamma := \partial\Omega. \tag{2.1c}$$

We use the standard notations $H^k(\Omega)$, $\|\cdot\|_k$, $(\cdot,\cdot)_k$, $k \geq 0$ denote the usual Soblev space, the standard Soblev norm and inner product, respectively. Especially when $k=0$, $L^2(\Omega) = H^0(\Omega)$ denotes the usual Lebegsgue space. We also introduce the spaces

$$H_0^1(\Omega) = \Big\{\varphi \in H^1(\Omega); \quad \varphi = 0, \ \text{ on } \Gamma\Big\},$$

$$L_0^2(\Omega) = \Big\{\varphi \in L^2(\Omega); \quad \int_\Omega \varphi = 0\Big\}.$$

Next, we give the mixed variational form of (2.1). Find

$$(\mathbf{u},p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega),$$

satisfied

$$a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=(\mathbf{f},\mathbf{v}),\quad \forall \mathbf{v}\in H_0^1(\Omega)^2, \tag{2.2a}$$
$$b(\mathbf{u},q)=0,\qquad \forall q\in L_0^2(\Omega), \tag{2.2b}$$

or

$$\mathcal{L}(\mathbf{u},p;\mathbf{v},q)=a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)+b(\mathbf{u},q)=(\mathbf{f},\mathbf{v}),\quad \forall(\mathbf{v},q)\in H_0^1(\Omega)^2\times L_0^2(\Omega), \tag{2.3}$$

where

$$a(\mathbf{u},\mathbf{v})=\int_\Omega \nabla\mathbf{u}:\nabla\mathbf{v}d\Omega,\qquad b(\mathbf{u},q)=-\int_\Omega q\nabla\cdot\mathbf{u}d\Omega.$$

As we all know, the bilinear form $\mathcal{L}$ satisfies the following inf-sup condition [17]

$$\inf_{(\mathbf{u},p)\in H_0^1(\Omega)^2\times L_0^2(\Omega)}\ \sup_{(\mathbf{v},q)\in H_0^1(\Omega)^2\times L_0^2(\Omega)} \frac{\mathcal{L}(\mathbf{u},p;\mathbf{v},q)}{(|\mathbf{u}|_1+\|p\|_0)(|\mathbf{v}|_1+\|q\|_0)}\geq\beta_c>0. \tag{2.4}$$

This promises that problem (2.3) has a unique solution.

In order to define a finite element method for the Stokes problem (2.1), we first introduce $\mathcal{J}_h$, which represents a family of triangulations of Ω, such that

1. the intersection of two different elements is at most a vertex or a whole edge,
2. the ratio of the diameter of any element in $\mathcal{J}_h$ to the diameter of its inscribed circle is bounded by a constant independent of h.

Let $\mathcal{P}_0$, $\mathcal{P}_1$ denote the space of constant polynomials and the linear polynomials space, put

$$\begin{aligned}
&R_0=\{q_h\in L^2(\Omega):\ \ \forall T\in\mathcal{J}_h,\ \varphi|_T\in\mathcal{P}_0\},\\
&P_1=\{\varphi\in C(\overline{\Omega}):\ \ \forall T\in\mathcal{J}_h,\ \varphi|_T\in\mathcal{P}_1\},\\
&X_h=\{P_1\cap H_0^1(\Omega)\}^2,\\
&Y_h=P_1\cap L_0^2(\Omega).
\end{aligned}$$

Then we can give P_1-P_1 discretization of (2.1). Find $(\mathbf{u}_h,p_h)\in X_h\times Y_h$,

$$\mathcal{L}(\mathbf{u}_h,p_h;\mathbf{v}_h,q_h)=(\mathbf{f},\mathbf{v}_h),\quad \forall(\mathbf{v}_h,q_h)\in X_h\times Y_h. \tag{2.5}$$

It is known that this (2.5) is not stable and can't be used directly. In [3] it represents an stabilization approach, that is to add $-G(p,q)$ to the left side of (2.2b). Find $(\tilde{\mathbf{u}},\tilde{p})\in H_0^1(\Omega)^2\times L_0^2(\Omega)$,

$$a(\tilde{\mathbf{u}},\mathbf{v})+b(\mathbf{v},\tilde{p})=(\mathbf{f},\mathbf{v}),\quad \forall\mathbf{v}\in H_0^1(\Omega)^2, \tag{2.6a}$$
$$b(\tilde{\mathbf{u}},q)-G(\tilde{p},q)=0,\qquad \forall q\in L_0^2(\Omega), \tag{2.6b}$$

where

$$G(\tilde{p},q)=\int_{\Omega}(\tilde{p}-\Pi\tilde{p})(q-\Pi q)\mathrm{d}\Omega. \tag{2.7}$$

Here the operator $\Pi : L^2(\Omega)\mapsto R_0$ has a piecewise constant range. And we assume that Π is continuous as an operator $L^2(\Omega)\mapsto L^2(\Omega)$:

$$\|\Pi p\|_0 \le C\|p\|_0. \tag{2.8}$$

Eqs. (2.6) are equivalent to the following form: find $(\tilde{\mathbf{u}},\tilde{p})\in H_0^1(\Omega)^2\times L_0^2(\Omega)$,

$$\tilde{\mathcal{L}}(\tilde{\mathbf{u}},\tilde{p};\mathbf{v},q)=(\mathbf{f},\mathbf{v}),\quad \forall(\mathbf{v},q)\in H_0^1(\Omega)^2\times L_0^2(\Omega), \tag{2.9}$$

where

$$\tilde{\mathcal{L}}(\tilde{\mathbf{u}},\tilde{p};\mathbf{v},q)=a(\tilde{\mathbf{u}},\mathbf{v})+b(\mathbf{v},\tilde{p})+b(\tilde{\mathbf{u}},q)-G(\tilde{p},q). \tag{2.10}$$

We can obtain the stabilization method by restricting (2.6) or (2.9) to the finite element spaces. That is: find $(\mathbf{u}_h,p_h)\in X_h\times Y_h$, such that

$$a(\mathbf{u}_h,\mathbf{v}_h)+b(\mathbf{v}_h,p_h)=(\mathbf{f},\mathbf{v}_h),\quad \forall\mathbf{v}_h\in X_h, \tag{2.11a}$$

$$b(\mathbf{u}_h,q_h)-G(p_h,q_h)=0,\qquad \forall q_h\in Y_h, \tag{2.11b}$$

or

$$\tilde{\mathcal{L}}(\mathbf{u}_h,p_h;\mathbf{v}_h,q_h)=(\mathbf{f},\mathbf{v}_h),\quad \forall(\mathbf{v}_h,q_h)\in X_h\times Y_h. \tag{2.12}$$

It has been proved that (2.12) is a stable variational problem in [3] because the inf-sup condition is available with the assumption (2.8)

$$\sup_{(\mathbf{v}_h,q_h)\in X_h\times Y_h}\frac{\tilde{\mathcal{L}}(\mathbf{u}_h,p_h;\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_1+\|q_h\|_0}\ge C\Big(\|\mathbf{u}_h\|_1+\|p_h\|_0\Big),\quad \forall(\mathbf{v}_h,q_h)\in X_h\times Y_h. \tag{2.13}$$

In addition after we assume that

$$\|(I-\Pi)p\|_0\le Ch\|p\|_1,\quad \forall p\in H^1(\Omega),$$

we can have the error estimates [3]

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_1+\|p-p_h\|_0\\ \le& C\Big(\inf_{q_h\in Y_h}\|p-q_h\|_0+\inf_{\mathbf{v}_h\in X_h}\|\mathbf{u}-\mathbf{v}_h\|_1+\|(I-\Pi)p\|_0\Big),\end{aligned} \tag{2.14}$$

where $(\mathbf{u},p)$ is the solution of the Stokes problem (2.1), $(\mathbf{u}_h,p_h)$ is the solution of the stabilized mixed problem (2.12).

We define

$$\mathbf{e}_h=\mathbf{u}-\mathbf{u}_h,\qquad \varepsilon_h=p-p_h,$$

and let P_0 denote the L^2-projection onto P_1 and let I_h denote the standard pointwise interpolation operator by elements of P_1.

3 Error indicators

We represent two types of error indicators, one is linked to the stabilization method, the other is based on the residual of the finite element discretization.

1. **The first kind of indicators**

We first define an indicator η_Π related to $G(p_h, q_h)$ by

$$\eta_\Pi = \|(I-\Pi)p_h\|_0. \tag{3.1}$$

As $(\mathbf{u}_h, p_h)$ is the discrete solution of the Stokes problem, this indicator will be easy to calculate.

2. **The second kind of indicators**

For every $T \in \mathcal{J}_h$, we define η_T as the following form:

$$\eta_T = \Big\{ h_T^2 \|P_0\mathbf{f} - \nabla p_h\|_{0,T}^2 + \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} h_E \Big\| \Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J \Big\|_{0,E}^2 + \|\nabla\cdot\mathbf{u}_h\|_{0,T}^2 \Big\}^{\frac{1}{2}}, \tag{3.2}$$

where the $[\cdot]_J$ denotes the jump of $(\cdot)$ across E.

Here we use the full discrete error,

$$|\mathbf{u}-\tilde{\mathbf{u}}|_1 + \|p-\tilde{p}\|_0 + |\tilde{\mathbf{u}}-\mathbf{u}_h|_1 + \|\tilde{p}-p_h\|_0. \tag{3.3}$$

Our aim is to prove the quantity

$$\eta_\Pi + \Big(\sum_{T\in\mathcal{J}_h} \eta_T^2\Big)^{\frac{1}{2}}, \tag{3.4}$$

is equivalent to the full error.

Theorem 3.1. *The a posteriori error estimate*

$$\begin{aligned} &|\mathbf{u}-\tilde{\mathbf{u}}|_1 + \|p-\tilde{p}\|_0 + |\tilde{\mathbf{u}}-\mathbf{u}_h|_1 + \|\tilde{p}-p_h\|_0 \\ \le& C\Big\{\eta_\Pi + \Big(\sum_{T\in\mathcal{J}_h} \eta_T^2 + h_T^2\|\mathbf{f}-P_0\mathbf{f}\|_{0,T}^2\Big)^{\frac{1}{2}}\Big\}, \end{aligned} \tag{3.5}$$

holds, where C only depends on Ω and the smallest angle in the triangulation $\mathcal{J}_h$.

Proof. We divide the full error into two parts:

$$|\mathbf{u}-\tilde{\mathbf{u}}|_1 + \|p-\tilde{p}\|_0 \quad \text{and} \quad |\tilde{\mathbf{u}}-\mathbf{u}_h|_1 + \|\tilde{p}-p_h\|_0,$$

and prove the inequality in two steps.

First, we just consider $|\mathbf{u}-\tilde{\mathbf{u}}|_1 + \|p-\tilde{p}\|_0$. We will make use of the property of $b(\cdot,\cdot)$. As $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\sup_{\mathbf{v}\in H_0^1(\Omega)^2} \frac{b(\mathbf{v},q)}{|\mathbf{v}|_1} \ge \beta\|q\|_0, \quad \forall q\in L_0^2. \tag{3.6}$$

Eq. (2.6b) has a unique solution $\boldsymbol{\omega} \in H_0^1(\Omega)^2$, so that

$$b(\boldsymbol{\omega}, q) = G(\tilde{p}, q), \quad \forall q \in L_0^2, \tag{3.7a}$$

$$|\boldsymbol{\omega}|_1 \le \beta^{-1}\|(I-\Pi)\|\|(I-\Pi)\tilde{p}\|_0. \tag{3.7b}$$

Thus $\tilde{\mathbf{u}} - \boldsymbol{\omega}$ and $\mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}$ belong to the space

$$V_0 = \big\{\mathbf{v} \in H_0^1(\Omega)^2;\ \forall q \in L_0^2(\Omega),\ b(\mathbf{v}, q) = 0\big\}.$$

By using the ellipticity property of $a(\cdot,\cdot)$, that is

$$a(\mathbf{v},\mathbf{v}) \ge \alpha|\mathbf{v}|_1^2, \quad \forall \mathbf{v} \in V_0,$$

we can derive that

$$\begin{aligned}
\alpha|\mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}|_1^2 \le & a(\mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}, \mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}) \\
= & -b(\mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}, p - \tilde{p}) + a(\boldsymbol{\omega}, \mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}) \\
= & a(\boldsymbol{\omega}, \mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}).
\end{aligned}$$

Thus we obtain the following two inequalities:

$$|\mathbf{u} - \tilde{\mathbf{u}} + \boldsymbol{\omega}|_1 \le C\alpha^{-1}|\boldsymbol{\omega}|_1, \tag{3.8a}$$

$$|\mathbf{u} - \tilde{\mathbf{u}}|_1 \le |\tilde{\mathbf{u}} - \mathbf{u} - \boldsymbol{\omega}|_1 + |\boldsymbol{\omega}|_1 \le (1 + C\alpha^{-1})\beta^{-1}\|(I-\Pi)\|\|(I-\Pi)\tilde{p}\|_0. \tag{3.8b}$$

Again we use the inf-sup condition of $b(\cdot,\cdot)$,

$$\beta\|p - \tilde{p}\|_0 \le \sup_{\mathbf{v}\in H_0^1(\Omega)^2} \frac{b(\mathbf{v}, p - \tilde{p})}{|\mathbf{v}|_1} = \sup_{\mathbf{v}\in H_0^1(\Omega)^2} \frac{-a(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v})}{|\mathbf{v}|_1} \le C|\mathbf{u} - \tilde{\mathbf{u}}|_1. \tag{3.9}$$

From (3.8) and (3.9), we can derive that

$$\begin{aligned}
& |\mathbf{u} - \tilde{\mathbf{u}}|_1 + \|p - \tilde{p}\|_0 \\
\le & C\|(I-\Pi)\tilde{p}\|_0 \le C\Big(\|(I-\Pi)p_h\|_0 + \|(I-\Pi)(\tilde{p} - p_h)\|_0\Big).
\end{aligned} \tag{3.10}$$

Second, we estimate $|\tilde{\mathbf{u}} - \mathbf{u}_h|_1 + \|\tilde{p} - p_h\|_0$. In the proof we have to use these two inequalities

$$\|\mathbf{v} - I_h\mathbf{v}\|_{0,T} \le Ch_T|\mathbf{v}|_{1,T}, \tag{3.11a}$$

$$\|\mathbf{v} - I_h\mathbf{v}\|_{0,E} \le Ch_T^{\frac{1}{2}}|\mathbf{v}|_{1,E}. \tag{3.11b}$$

By subtracting (2.12) from (2.13) and using (2.6b), (3.11), we can deduce that

$$\begin{aligned}
& \tilde{\mathcal{L}}(\tilde{\mathbf{u}} - \mathbf{u}_h, \tilde{p} - p_h; \mathbf{v}, q) \\
= & \big(\nabla(\tilde{\mathbf{u}} - \mathbf{u}_h), \nabla(\mathbf{v} - I_h\mathbf{v})\big) - \big(\tilde{p} - p_h, \nabla(\mathbf{v} - I_h\mathbf{v})\big) \\
& - \big(q, \nabla\cdot(\tilde{\mathbf{u}} - \mathbf{u}_h)\big) - G(\tilde{p} - p_h, q)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{T\in\mathcal{J}_h} \Big\{ -\big(\Delta(\tilde{\mathbf{u}}-\mathbf{u}_h), \mathbf{v}-I_h\mathbf{v}\big)_T + \big(\nabla(\tilde{p}-p_h), \mathbf{v}-I_h\mathbf{v}\big)_T \\
&\quad + \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} \Big(\Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J, \mathbf{v}-I_h\mathbf{v}\Big)_E \\
&\quad + (q, \nabla\cdot\mathbf{u}_h)_T + \big((I-\Pi)p_h, (I-\Pi)q\big)_T \Big\} \\
&\le \sum_{T\in\mathcal{J}_h} \Big\{ Ch_T\|P_0\mathbf{f}-\nabla p_h\|_{0,T}|\mathbf{v}|_1 + Ch_E^{\frac{1}{2}}\Big\|\Big[\frac{\partial \mathbf{u}_h}{\partial n} \\
&\quad - p_h\cdot n\Big]_J\Big\|_{0,E}|\mathbf{v}|_1 + Ch_T\|\mathbf{f}-P_0\mathbf{f}\|_{0,T}|v|_1 \\
&\quad + \|\nabla\cdot\mathbf{u}_h\|_{0,T}\|q\|_{0,T} + \|(I-\Pi)p_h\|_{0,T}\|I-\Pi\|\,\|q\|_{0,T} \Big\} \\
&\le C\Big\{\eta_\Pi + \Big[\sum_{T\in\mathcal{J}_h}\big(\eta_T^2 + h_T^2\|\mathbf{f}-P_0\mathbf{f}\|_{0,T}^2\big)\Big]^{\frac{1}{2}}\Big\}\big\{|\mathbf{v}|_{1,T} + \|q\|_{0,T}\big\}.
\end{aligned}
$$

Combining this inequality with (2.13), we can conclude that

$$
|\tilde{\mathbf{u}}-\mathbf{u}_h|_1 + \|\tilde{p}-p_h\|_0 \le C\Big\{\eta_\Pi + \Big[\sum_{T\in\mathcal{J}_h}\big(\eta_T^2 + h_T^2\|\mathbf{f}-P_0\mathbf{f}\|_{0,T}^2\big)\Big]^{\frac{1}{2}}\Big\}. \tag{3.12}
$$

From (3.10) and (3.12), we can derive the conclusion. $\square$

Theorem 3.2. *The following estimates hold,*

$$
\eta_\Pi \le C\Big(|\mathbf{u}-\tilde{\mathbf{u}}|_1 + \|\tilde{p}-p_h\|_0\Big), \tag{3.13a}
$$

$$
\eta_T \le C\Big(|\mathbf{u}-\tilde{\mathbf{u}}|_{1,T} + |\tilde{\mathbf{u}}-\mathbf{u}_h|_{1,T} + \|\tilde{p}-p_h\|_{0,T} + h_T\|\mathbf{f}-P_0\mathbf{f}\|_{0,T}\Big), \tag{3.13b}
$$

where C only depends on Ω and the smallest angle in the triangulation $\mathcal{J}_h$.

Proof. We will prove the two inequalities respectively.

1. Subtracting (2.6b) from (2.2b), we can obtain

$$
b(\mathbf{u}-\tilde{\mathbf{u}}, q) = -G(\tilde{p}, q), \quad \forall q\in L_0^2.
$$

As $G(\cdot,\cdot)$ has the following property

$$
G(q,q) = \|(I-\Pi)q\|_0^2, \quad \forall q\in L_0^2,
$$

when we choose $q=\tilde{p}$, we can easily get

$$
\|(I-\Pi)\tilde{p}\|_0 \le C|\mathbf{u}-\tilde{\mathbf{u}}|_1,
$$

where C only depends on $\|I-\Pi\|_0$ and the norm of $b(\cdot,\cdot)$.

Using a triangle inequality, we can conclude that

$$\eta_\Pi = \|(I-\Pi)p_h\|_0 \le C\big(|\mathbf{u}-\tilde{\mathbf{u}}|_1 + \|\tilde{p}-p_h\|_0\big).$$

2. In the proof of (3.13b), we will use the property of bubble function. Let $\lambda_{Ti}, i = 1,2,3$ denote the barycentric coordinates of $T \in \mathcal{J}_h$ and $E_{ij}, 1 \le i \le j \le 3$ denote the three edge of T with x_i and x_j as its endpoints. Give two bubble functions

$$\psi_T = \frac{\lambda_{T1}\lambda_{T2}\lambda_{T3}}{\int_T \lambda_{T1}\lambda_{T2}\lambda_{T3}} \quad \text{and} \quad \psi_{E_{ij}} = \frac{\lambda_{Ti}\lambda_{Tj}}{\int_{E_{ij}} \lambda_{Ti}\lambda_{Tj}},$$

define two functions on T

$$\begin{aligned}
\mathbf{w}_T =& \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} h_E^2\Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J \psi_E - \psi_T\Big\{|T|^2[P_0\mathbf{f}-\nabla p_h] \\
&+ \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} h_E^2\Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J \int_T \psi_E\Big\}, \qquad (3.14a)\\
q_T =& \nabla\cdot\mathbf{u}_h, \qquad (3.14b)
\end{aligned}$$

where $|T|$ denotes the area of T. It is obvious that

$$\begin{aligned}
&\int_T \mathbf{w}_T = -|T|^2(P_0\mathbf{f}-\nabla p_h),\\
&\int_E \mathbf{w}_T = \frac{1}{2}h_E^2\Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J, \quad \forall E\subset\partial T\cap\Omega.
\end{aligned}$$

With these properties we can get

$$\begin{aligned}
\eta_T^2 =& h_T^2\|P_0\mathbf{f}-\nabla p_h\|_{0,T}^2 + \frac{1}{2}\sum_{E\in\partial T\cap\Omega} h_E\Big\|\Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J\Big\|_{0,E}^2 + \|\nabla\cdot\mathbf{u}_h\|_{0,T}^2\\
=& -(P_0\mathbf{f}-\nabla p_h, \mathbf{w}_T)_T + \sum_{EE\in\partial T\cap\Omega}\Big(\Big[\frac{\partial \mathbf{u}_h}{\partial n} - p_h\cdot n\Big]_J, \mathbf{w}_T\Big)_E + (\nabla\cdot\mathbf{u}_h, q_T)_T\\
=& -(P_0\mathbf{f}-\nabla p_h, \mathbf{w}_T)_T + \big(\triangle(-\tilde{\mathbf{u}}+\mathbf{u}_h)_T - (-\tilde{p}+p_h), \mathbf{w}_T\big)_T\\
&+ \big(\nabla(-\tilde{\mathbf{u}}+\mathbf{u}_h)_T - (-\tilde{p}+p_h), \nabla\mathbf{w}_T\big)_T + \big(\nabla\cdot(\mathbf{u}_h-\mathbf{u}), q_T\big)_T\\
=& (\mathbf{f}-P_0\mathbf{f}, \mathbf{w}_T)_T + \big(\nabla(-\tilde{\mathbf{u}}+\mathbf{u}_h), \nabla\mathbf{w}_T\big)_T + (\tilde{p}-p_h, \nabla\mathbf{w}_T)_T\\
&+ \big(\nabla\cdot(\mathbf{u}_h-\tilde{\mathbf{u}}), q_T\big)_T + \big(\nabla\cdot(\tilde{\mathbf{u}}-\mathbf{u}), q_T\big)_T\\
\le& C\big\{|\tilde{\mathbf{u}}-\mathbf{u}_h|_{1,T} + \|\tilde{p}-p_h\|_{0,T} + |\mathbf{u}-\tilde{\mathbf{u}}|_{1,T}\big\}\big\{\|q_T\|_{0,T} + |\mathbf{w}_T|_{1,T}\big\}\\
&+ \|\mathbf{f}-P_0\mathbf{f}\|_{0,T}\|\mathbf{w}_T\|_{0,T}. \qquad (3.15)
\end{aligned}$$

Note that

$$\|q_T\|_{0,T} \le \eta_T, \quad |\mathbf{w}|_{1,T} \le C\eta_T, \quad \|\mathbf{w}_T\|_{0,T} \le Ch_T\eta_T, \qquad (3.16)$$

thus we can obtain the conclusion by combining (3.15) and (3.16). □

4 Numerical examples

In our numerical experiments, we use the usual following adaptive algorithm

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Refine}.$$

Concretely we solve our problem in the following strategy:

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1. Give an initial triangulation $\mathcal{J}_0$ and a tolerance η^*. Solve the problem on this triangulation,
2. Compute
$$\eta_\Pi + \Big(\sum_{T\in\mathcal{J}_h} \eta_T^2\Big)^{\frac{1}{2}};$$
if it is less than η^* we get the final solution and stop. Otherwise, go to Step 3,
3. Compute $\eta_{\Pi,T} + \eta_T$ and their mean value. Generate a new mesh size h by the method presented in [18] and go back to Step 2.

The circle of Step 2 and Step 3 is iterated 4–5 times in our computation.

We will present two examples in this section to show the efficiency of our error estimators in the process of constructing self-adaptive meshes and in estimating the discretization errors.

Example 4.1. We consider a driven cavity problem in the domain $\Omega = (0,1) \times (0,1)$, which means that $u_x = 1$, $u_y = 0$ on the upper side and $\mathbf{u} = \mathbf{0}$ on the other three sides. We start from the initial mesh in Fig. 1 and after 5 steps get Fig. 2, from which we can see that at the two top corners there are more triangles than other areas. In Figs. 5 and 6, we present the velocity field in uniform mesh and adaptive form mesh after 2 steps given in Figs. 3 and 4, with nearly the same number of triangles. From these figures we can see that the solution using the a posteriori error analysis gives a more accurate approximation to the real situation.

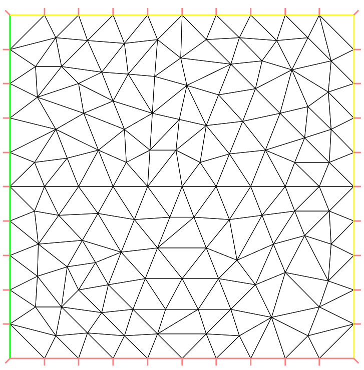

Figure 1: Example 4.1: Initial mesh.

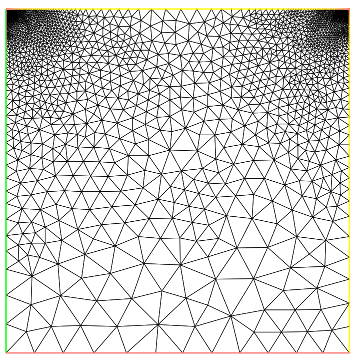

Figure 2: Example 4.1: the mesh after 5 steps.

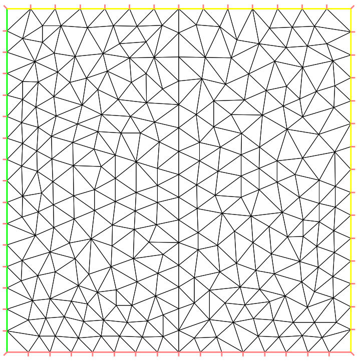

Figure 3: Example 4.1: Uniform mesh.

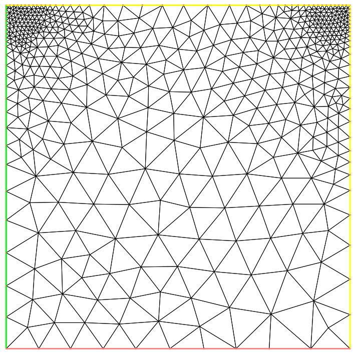

Figure 4: Example 4.1: Adaptive mesh after 2 steps.

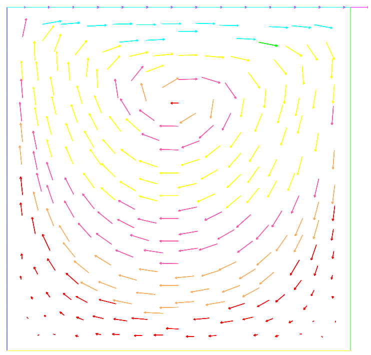

Figure 5: Example 4.1: Velocity field in uniform mesh of Fig. 3.

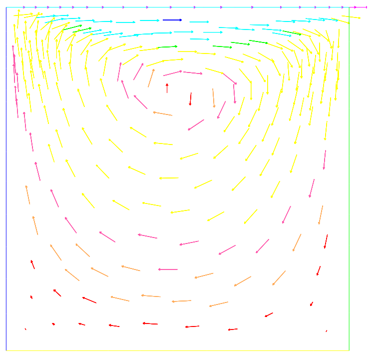

Figure 6: Example 4.1: Velocity field in adaptive mesh of Fig. 4.

Example 4.2. We consider a problem [19] with a smooth solutions, which are given by

$$\begin{aligned} u_x &= 1.5r^{\frac{1}{2}}\big(\cos(0.5\theta) - \cos(1.5\theta)\big), \\ u_y &= 1.5r^{\frac{1}{2}}\big(3\sin(0.5\theta) - \sin(1.5\theta)\big), \\ p &= -6r^{-\frac{1}{2}}\cos(0.5\theta), \end{aligned}$$

in a circular domain with radius 1 and angle 2π, and with a non-homogeneous Dirichlet boundary conditions on the curved part of the boundary and homogeneous Dirichlet boundary conditions on the straight part of the boundary. We start the strategies from the initial triangulations, as in Fig. 7 and refine 3 times shown in the following three figures. It is observed that in the noncontinuous area, there are much more elements than these in the continuous area. In Table 1, we present the ratio of the error indicators and the discrete error, which is defined as the effective index in [20]. Here N is the number of element in the triangulations. At the same time, we compare the $|\mathbf{e}_h|_1/|u|_1$ and $\|\varepsilon_h\|_0/\|p\|_0$ in with adaptive mesh and uniform mesh. In the last two columns, we give $|\bar{\mathbf{e}}_h|_1/|u|_1$ and $\|\bar{\varepsilon}_h\|_0/\|p\|_0$ as the discretization errors in uniform meshes. From the result, we can obviously see the advantage of adaptive mesh.

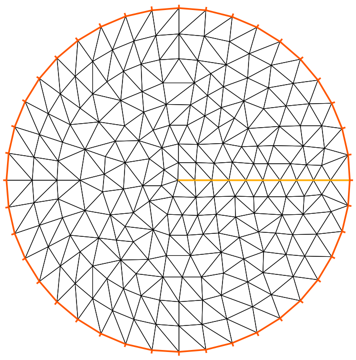

Figure 7: Example 4.2: initial mesh.

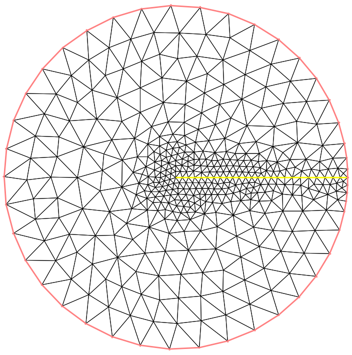

Figure 8: Example 4.1: mesh after step 1.

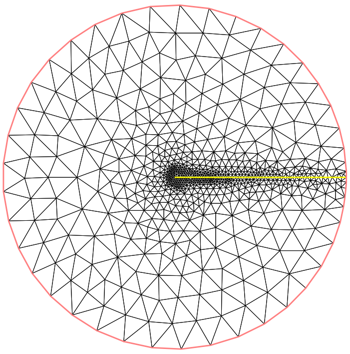

Figure 9: Example 4.2: mesh after step 2.

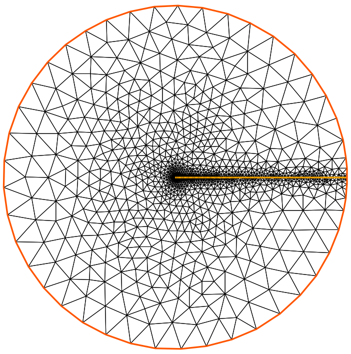

Figure 10: Example 4.2: mesh after step 3.

Table 1: The errors and the effective index.

Mesh	N	ratio	$\frac{\|\mathbf{e}_h\|_1}{\|u\|_1}$	$\frac{\|\varepsilon_h\|_0}{\|p\|_0}$	N	$\frac{\|\bar{\mathbf{e}}_h\|_1}{\|u\|_1}$	$\frac{\|\bar{\varepsilon}_h\|_0}{\|p\|_0}$
1	805	2.87335	0.180503	0.239083	795	0.236052	0.34612
2	1490	3.32213	0.126404	0.146115	1486	0.214606	0.307636
3	2848	3.74112	0.0884136	0.0928755	2889	0.183395	0.257435

Acknowledgements

The research is supported by the NSF of China (No. 10971166) and the National Basic Research Program (No. 2005CB321703).

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