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Error Analysis for a Non-Monotone FEM for a Singularly Perturbed Problem with Two Small Parameters

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Abstract. In this paper, we consider a singularly perturbed convection-diffusion problem. The problem involves two small parameters that gives rise to two boundary layers at two endpoints of the domain. For this problem, a non-monotone finite element methods is used. A priori error bound in the maximum norm is obtained. Based on the a priori error bound, we show that there exists Bakhvalov-type mesh that gives optimal error bound of $\mathcal{O}(N^{-2})$ which is robust with respect to the two perturbation parameters. Numerical results are given that confirm the theoretical result.

AMS subject classifications: 65N30

Key words: Non-monotone, singularly perturbed, convection-diffusion, Bakhvalov-type, two small parameters.

1 Introduction

Singularly perturbed problems arise from many branches of engineering and applied mathematics, such as fluid dynamic, quantum mechanics, chemical reactor, gas porous electrodes theory, etc. It is well known that the exact solution to these problems have thin layer(s), where the solution vary very rapidly, while away from the layer(s) the solution vary very slowly. For this reason, some standard numerical methods on a uniform mesh will not give satisfactory numerical results. Thus, to obtain some reliable numerical results to these problems, piecewise uniform mesh, so-called Shishkin mesh and Bakhvalov mesh were often used; see, e.g., [1–6]. But for singularly perturbed problems with two small parameters only few results were studied in the literature.

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In this paper, we consider the following singularly perturbed convection-diffusion problem with two small parameters

$$Lu := -\varepsilon_d u'' + \varepsilon_c b u' + cu = f, \quad x \in (0,1), \tag{1.1a}$$

$$u(0) = 0, \quad u(1) = 0, \tag{1.1b}$$

where $0 < \varepsilon_d \ll 1$ and $0 \le \varepsilon_c \ll 1$ are two small parameters, while $b \in C^1(0,1)$ and $c, f \in C(0,1)$ and there exists a constant ϱ such that

$$c(x) \ge \varrho > 0, \quad x \in [0,1]. \tag{1.2}$$

Under these assumptions, the above problem (1.1)-(1.2) possesses a unique solution with two boundary layers at both end points depending on the values of the parameters ε_d and ε_c.

Linß and Roos [7] considered the problem (1.1a)-(1.1b) and obtained uniform convergence of $\mathcal{O}(N^{-1}\ln N)$ by using a first-order upwind difference scheme on a piecewises uniform Shishkin mesh. In [8], a almost second order monotone numerical method combined with a piecewise- uniform Shishkin mesh was developed for problem (1.1a)-(1.1b). Linß [9] used a streamline-diffusion FEM (SDFEM) to solve the problem (1.1a)-(1.1b), numerically. Then, he derived a posteriori error estimate in the maximum norm for the SDFEM. At last, the a posteriori error bound was used to design an adaptived method. Roos and Uzelac [10] obtained almost second-order parameters uniform convergence (in the maximum norm) by the SDFEM on a Shishkin mesh. For problem (1.1a)-(1.1b), Wu et al. [11] presented an adaptived moving method which was first-order convergent uniformly in both perturbation parameters.

In this paper, we will use some techniques presented in [6] to study the optimal rate of convergence by using a non-monotone FEM to discretize problem (1.1a)-(1.1b) on a Bakhvalov mesh. By using the Green's function and the bounds on the L_1-norm of G and G'', a second order uniform convergence (in the maximum norm) is obtained. Finally, one numerical experiment is proposed to support the theoretical result.

Throughout this paper, C denote a generic constant that is independent of the parameters ε_d, ε_c and N, the number of mesh points. It may take different values in different places.

2 Discretization

In this section, we will describe the discretization method for problem (1.1a)-(1.1b), and give the bounds of its solution and its derivatives.

The weak formulation of (1.1a)-(1.1b): Find $u \in H_0^1(0,1)$ to satisfy

$$a(u,v) := \varepsilon_d (u',v') + \varepsilon_c (bu',v) + (cu,v) = (f,v), \quad \forall v \in H_0^1(0,1), \tag{2.1}$$

where $(\cdot,\cdot)$ is the standard inner product in $L_2(0,1)$.