Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 2, pp. 324-348

DOI: 10.4208/aamm.2015.m1326
April 2017

Robust Semi-Discrete and Fully Discrete Hybrid Stress Finite Element Methods for Elastodynamic Problems

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Received 18 October 2015; Accepted (in revised version) 17 February 2016

Abstract. This paper analyzes semi-discrete and fully discrete hybrid stress quadrilateral finite element methods for 2-dimensional linear elastodynamic problems. The methods use a 4 node hybrid stress quadrilateral element in the space discretization. In the fully discrete scheme, an implicit second-order scheme is adopted in the time discretization. We derive optimal a priori error estimates for the two schemes and an unconditional stability result for the fully discrete scheme. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N12, 65N15, 65N30

Key words: Elastodynamic problem, hybrid stress finite element, semi-discrete, fully discrete, error estimate.

1 Introduction

Consider the following 2-dimensional linear elastodynamic problems:

$$\begin{cases} \mathbf{u}_{tt} - \mathrm{div}\boldsymbol{\sigma} = \mathbf{f}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [0,T], \\ \boldsymbol{\sigma} = \mathbb{C}\epsilon(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\mathbf{I}, & (\mathbf{x},t) \in \Omega \times [0,T], \\ \mathbf{u} = 0, & (\mathbf{x},t) \in \Gamma \times [0,T], \\ \mathbf{u}(\mathbf{x},0) = \boldsymbol{\varphi}_0(\mathbf{x}), \quad \mathbf{u}_t(\mathbf{x},0) = \boldsymbol{\varphi}_1(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^2$ is a convex and bounded open set with boundary Γ, T a positive constant, $\mathbf{u} = (u_1, u_2)^T$ the displacement field, $\boldsymbol{\sigma} = (\sigma_{ij})_{2\times 2}$ the symmetric stress tensor, $\epsilon(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ the strain tensor, and $\mathbf{f}(\mathbf{x},t)$ the body force. $\lambda, \mu > 0$ are the Lamé parameters. $\boldsymbol{\varphi}_0(\mathbf{x})$, $\boldsymbol{\varphi}_1(\mathbf{x})$ are initial data. We assume that $\mathbf{f}$, $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_1$ are as regular as necessary.

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Makridakis [1] analyzed semi-discrete and fully discrete mixed finite element methods for linear elastodynamics, where in the fully-discrete scheme rational approximations to the cosine were used for the time discretization. In [2], Bécache, Joly and Tsogka changed the two-order system (1.1) to a first order system, and constructed a new family of mixed finite elements, which allow mass lumping and lead to efficiency computation once explicit time discretization is made. Boulaajine, Farhloul and Paquet [3, 4] proposed a new dual mixed finite element method to solve the linear elastodynamic system with explicit/implicit Newmark schemes for the time discretizaton. In [5], Lai, Huang and Chen established a C^0-continuous time stepping displacement-type finite element method. Hughes and Hulbert [6] developed space-time finite element methods which use the discontinuous Galerkin method in time and incorporate stabilizing terms of least-squares type. Idesman [7] presented space-time finite element methods on structured and unstructured meshes based on time continuous and discontinuous Galerkin methods. In [8], Cheng and Xie considered a space-time nonconforming finite element method.

For static elasticity problems, the hybrid stress finite element method pioneered by Pian [9] is known to be an efficient approach to improve the performance of the standard 4-node compatible displacement quadrilateral (bilinear) element [10–14]. The method is based on the Hellinger-Reissner variational principle which includes unknowns of displacements and stresses, and leads to a displacement-type scheme after locally eliminating stress parameters. In [10] Pian and Sumihara proposed a robust 4-node hybrid stress quadrilateral element (abbr. PS element) by using isoparametric bilinear displacement approximations and a rational choice of 5-parameter stress mode. Yu, Xie and Carstensen [13] proved that the PS element is uniformly convergent on certain meshes as the Lamé constant $\lambda \rightarrow \infty$.

By using PS element for the space discretization, Yu and Xie [15] presented semi-discrete and fully discrete hybrid stress quadrilateral finite element methods for the linear elastodynamic problems (1.1), where a second-order center difference is used for the time discretization in the fully discrete scheme. Error estimates of the two schemes, as well as a conditional stability result for the fully discrete scheme, were derived there with the constant factors in all the estimates depending on λ.

In this paper, we shall further study the semi-discrete hybrid stress scheme proposed in [15] and develop a new fully discrete scheme by using an implicit second-order scheme in the time discretization. Our analysis and new scheme improve [15] in the following aspects.

- The constant factors in the derived optimal error estimates of the semi-discrete and fully discrete schemes are uniform with respect to the Lamé constant λ.
- The new fully discrete scheme is unconditionally stable.

The rest of the paper is organized as follows. Section 2 introduces notations and weak formulations. Section 3 is devoted to the error estimation of the semi-discrete hybrid