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Analysis of Solving Galerkin Finite Element Methods with Symmetric Pressure Stabilization for the Unsteady Navier-Stokes Equations Using Conforming Equal Order Interpolation

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Abstract. This paper gives analysis of a semi-discrete scheme using equal order interpolation to solve unsteady Navier-Stokes equations. A unified pressure stabilized term is added to our scheme. We proved the uniform error estimates with respect to the Reynolds number, provided the exact solution is smooth.

AMS subject classifications: 65M60, 65N30

Key words: Unsteady Navier-Stokes equations, symmetric pressure stabilization, equal order interpolation.

1 Introduction

Let $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ be a Lipschitz-continuous polyhedral region with boundary $\partial\Omega$ and I be a fixed positive constant. We consider the following time-dependent incompressible Navier-Stokes equations with homogeneous boundary conditions:

$$\begin{cases} \boldsymbol{u}_t+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f} & \text{in } \Omega\times[0,I],\\ \nabla\cdot\boldsymbol{u}=0 & \text{in } \Omega\times[0,I],\\ \boldsymbol{u}=\boldsymbol{0} & \text{on } \partial\Omega\times[0,I],\\ \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},t)\in\mathbb{R}^d\times[0,I]$ denotes the velocities, $p=p(\boldsymbol{x},t)\in\mathbb{R}\times[0,I]$ denotes the pressure, $\boldsymbol{f}=\boldsymbol{f}(\boldsymbol{x},t)\in\mathbb{R}^d\times[0,I]$ denotes the body forces and $\boldsymbol{u}_0=\boldsymbol{u}_0(\boldsymbol{x})\in\mathbb{R}^d$ stands for the initial velocities, $\nu=Re^{-1}$ denotes the viscosity coefficient, Re denotes the Reynolds number.

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The mixed finite element method is an important method to solve the Navier-Stokes equations (NSEs). As is well known, there are two major difficulties in solving NSEs using standard Galerkin method. The first one is the approximate finite element spaces for velocity and pressure must satisfy the inf-sup stability condition, since it cannot be inherited from the continuous problem. The second one is even with inf-sup stable elements, the numerical solution may become unstable when the viscosity is small. Over the past decades, much work has been devoted to studying the stabilized mixed methods to solve those two issues: the streamline diffusion (SD) method [1–10], the variational multiscale method (VMS) method [11–15], the orthogonal subsclae method [16], the continuous interior penalty (CIP) method [17], the local projection stabilized (LPS) method [18,19] and the multi-level stabilized methods [35]. More work about the pressure stabilized methods can be found in [20–22,32–34]. Among these work for unsteady NSEs, the analysis are usually different in two cases: (1) use of an inf-sup stable velocity pressure pair; (2) use of equal order interpolation for velocities and pressure.

In both cases effects due to dominating convection must be stabilized. For the first case, we refer to the VMS method [8], where the authors present finite element error estimates of a VMS method for the unsteady incompressible NSEs. The constants in these estimates do not depend on the Reynolds number but on a reduced Reynolds number or on the mesh size of a coarse mesh. For the second case, pressure stabilization technique must be involved. We refer to SD [3], CIP [13] and LPS [16] methods, where same techniques are used to stabilize pressure and effects due to dominating convection. Error estimates uniformly with the Reynolds number are obtained, provided the exact solution is smooth. We should aware of that, in the analysis of unsteady NSEs, the "true" uniform error estimates (error estimates only dependent on the force term) with respect to Reynolds number haven't appeared yet. The analysis we mentioned above are dependent on assuming the exact solution is smooth enough, so is the case in this paper.

In this paper, we prove the conforming equal order elements only with pressure stabilized strategy also have uniform error estimates with respect to the Reynolds number, provided the exact solution is smooth. Numerical performance are implemented to confirm and illustrate our theoretical analysis.

An outline of the paper is as follows. In Section 2, we introduce necessary notations. In Section 3 we propose our unified pressure stabilized method. In Section 4 we give the analysis of stability and error estimates for our method. In Section 5, we give numerical experiments to confirm our analysis.

Throughout this paper, we use C to denote a positive constant independent of Δt, h and ν, not necessarily the same at each occurrence. The notation $a \lesssim b$ represents $a \leq Cb$.

2 Basic notations

For any bounded domain $\Lambda \subset \mathbb{R}^d$, let $H^m(\Lambda)$ and $H_0^m(\Lambda)$ denote the usual m^{th}-order Sobolev spaces on Λ, and $\|\cdot\|_{m,\Lambda}$, $|\cdot|_{m,\Lambda}$ denote the norm and semi-norm on these spaces.