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# A High-Accuracy Mechanical Quadrature Method for Solving the Axisymmetric Poisson's Equation

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**Abstract.** In this article, we consider the numerical solution for Poisson's equation in axisymmetric geometry. When the boundary condition and source term are axisymmetric, the problem reduces to solving Poisson's equation in cylindrical coordinates in the two-dimensional $(r,z)$ region of the original three-dimensional domain $S$. Hence, the original boundary value problem is reduced to a two-dimensional one. To make use of the Mechanical quadrature method (MQM), it is necessary to calculate a particular solution, which can be subtracted off, so that MQM can be used to solve the resulting Laplace problem, which possesses high accuracy order $\mathcal{O}(h^3_{\max})$ and low computing complexities. Moreover, the multivariate asymptotic error expansion of MQM accompanied with $\mathcal{O}(h^3_i)$ for all mesh widths $h_i$ is got. Hence, once discrete equations with coarse meshes are solved in parallel, the higher accuracy order of numerical approximations can be at least $\mathcal{O}(h^5_{\max})$ by the splitting extrapolation algorithm (SEA). Meanwhile, a posteriori asymptotic error estimate is derived, which can be used to construct self-adaptive algorithms. The numerical examples support our theoretical analysis.

**AMS subject classifications**: 65N38, 65R20

**Key words**: Mechanical quadrature method, splitting extrapolation algorithm, Poisson's equation, a posteriori estimate.

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## 1 Introduction

The Poisson equation is a basic equation for many areas of science and engineering, such as electrostatic and gravitational potential theories, astronomy, optics, fluid dynamics, steady-state heat flow and computer graphics [1–5]. There were many numerical models for solving the 2D and 3D Poisson equation [6–10]. Finite element methods (FEM) was

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proposed to solving the axisymmetric Poisson equation on polygonal domains [26]. In this paper, we consider the numerical solution of the axisymmetric Poisson's equation

$$\begin{cases} \Delta u(P)=f(P), & P\in V, \\ u(P)=g(P), & P\in S, \end{cases} \tag{1.1}$$

where $V$ is an axisymmetric bounded domain of space $\Re^3$, and formed by rotating a two-dimensional bounded region $\Omega$ with the boundary $\Gamma=\cup_{m=1}^{d}\Gamma_m\in\Re^2$, around the $z$-axis, called a generatrix line of $V$, and $S$ is the surface of $V$. The function $f(P)$ is a smooth source term that is axisymmetric, and the boundary condition is also axisymmetric.

When we get a particular solution $u_p(P)$ of (1.1). Let $v(P)=u(P)-u_p(P)$, and $v(P)$ satisfies the boundary value problem

$$\begin{cases} \Delta v(P)=0, & P\in V, \\ v(P)=g(P)-u_p(P), & P\in S. \end{cases} \tag{1.2}$$

By the potential theory, the solutions of (1.2) can be represented as a single-layer potential

$$v(Q)=\int_S G(Q,P)\rho(P)ds_P, \quad Q\in V, \tag{1.3}$$

where $G(Q,P)$ is the foundation solution of three-dimensional Laplace's equation. $\rho(P)$ is the solution of the following equation

$$v(Q)=\int_S G(Q,P)\rho(P)ds_P, \quad Q\in S. \tag{1.4}$$

Let

$$P=(r_P,0,z_P) \quad \text{and} \quad Q=(r_Q\cos\theta,r_Q\sin\theta,z_Q).$$

Then integrating $G(Q,P)$ over $\theta$, we get the axisymmetric fundamental solution

$$G_A(Q,P)=\frac{K(k)}{\pi R}, \tag{1.5}$$

where $K(k)$ is the complete elliptic integral of the first kind,

$$k^2=\frac{4r_P r_Q}{R^2}, \quad R=[(r_P+r_Q)^2+(z_P-z_Q)^2]^{1/2}. \tag{1.6}$$

Especially, when $r_P=0$, we have the following result

$$G_A(Q,P)=\frac{1}{2(r_Q^2+(z_Q-z_P)^2)^{1/2}}. \tag{1.7}$$

**Lemma 1.1.** *(1) When $r_P>0$, $G_A(Q,P)$ has logarithmic singularity; (2) When $r_P=0$, $G_A(Q,P)$ has Cauchy singularity.*