Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **3**, No. 1, pp. 109-120

DOI: 10.4208/aamm.10-m1027
February 2011

On Modifications of Continuous and Discrete Maximum Principles for Reaction-Diffusion Problems

István Faragó[1], Sergey Korotov[1,2,*] and Tamás Szabó[1]

[1] *Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, H–1117, Budapest, Pázmány P. s. 1/c., Hungary*
[2] *BCAM - Basque Center for Applied Mathematics, Bizkaia Technology Park, Building 500, E–48160 Derio, Basque Country, Spain*

Received 13 May 2010; Accepted (in revised version) 7 July 2010
Available online 15 October 2010

Abstract. In this work, we present and discuss some modifications, in the form of two-sided estimation (and also for arbitrary source functions instead of usual sign-conditions), of continuous and discrete maximum principles for the reaction-diffusion problems solved by the finite element and finite difference methods.

AMS subject classifications: 35B50, 65N06, 65N30, 65N50

Key words: Reaction-diffusion problem, maximum principle, discrete maximum principle, monotone matrix, two-sided a priori estimation, code validification.

---

1 Motivation

Consider the following boundary-value problem of elliptic type: find a function $u\in C^2(\overline{\Omega})$, such that

$$-\Delta u + cu = f, \quad \text{in } \Omega, \tag{1.1a}$$

$$u = 0, \qquad\qquad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega\subset\mathbf{R}^d$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$ and the reactive coefficient $c(x)\geq 0$ for all $x\in\overline{\Omega}$. We also assume that c and the right-hand side (be shortly called RHS, or source, in what follows) function f are both from $C(\overline{\Omega})$.

The classical solution of problem (1.1) is known to satisfy the so-called *maximum principle* (MP), which is often written as follows:

$$f(x) \geq 0, \quad \text{in } \overline{\Omega} \Longrightarrow \max_{x\in\overline{\Omega}} u(x) \geq 0. \tag{1.2}$$

---

*Corresponding author.
Email: faragois@cs.elte.hu (I. Faragó), smkorotov@gmail.com, korotov@bcamath.org (S. Korotov), szabot@cs.elte.hu (T. Szabó)

For more general forms of MPs (for example in case of nonhomogeneous Dirichlet or mixed boundary conditions, see e.g., [13,18]), however it is important to emphasize that MPs often considered for problems of elliptic type in the numerical community have a form of implication involving certain sign-type conditions only (like in (1.2)). The problem of construction (and proofs of validity) of suitable discrete analogues (the discrete maximum principles, or DMPs in short) of MPs for various types of numerical discretizations has also attracted a lot of attention by numerical scientists during last decades, see [4,6,7,13,16,21,24–26] and references therein.

However, from (1.2) we can only get an information on the sign of the unknown solution u, which can be often important to know (and provide on the discrete level in practical calculations especially if u models some physical quantity which is non-negative by definition-absolute temperature, density, concentration, etc. [17]). At the same time there exist various a priori (upper and lower) estimates (see e.g., [18] and references therein) on the magnitude of solutions of certain elliptic problems in the PDE community, which can also be of interest both theoretically and practically. Also, many real-life problems have function f which may easily change its sign in the solution domain, and, in addition, most of MPs used (and imitated in numerics) e.g., for parabolic equations are formulated independently of signs of the source functions, see [8] and references therein. The goal of this work is to combine several available theoretical estimates in order to get a priori two-sided bounds for the (classical) solutions of elliptic problems (1.1) (with positive reactive terms) for arbitrary source functions and show how to provide the validity of their natural discrete analogues if some popular numerical technique (e.g., the finite element method (FEM) or the finite difference method (FDM)) is used for the discretization.

First, we shall present the following key result on continuous level.

Theorem 1.1. *Let functions c and f in* (1.1) *be from $C(\overline{\Omega})$, and let, additionally,*

$$c(x) \geq c_0 > 0, \quad \textit{for all} \;\; x \in \overline{\Omega}. \tag{1.3}$$

Then the following (a priori) two-sided estimates for the classical solution of problem (1.1) *are valid:*

$$\min\Big\{0, \min_{s\in\overline{\Omega}} \frac{f(s)}{c(s)}\Big\} \leq u(x) \leq \max\Big\{0, \max_{s\in\overline{\Omega}} \frac{f(s)}{c(s)}\Big\}, \quad \textit{for any} \;\; x \in \overline{\Omega}. \tag{1.4}$$

Proof. To prove the upper estimate for u in above, one notices first that it is clearly valid if $u \leq 0$ everywhere in $\overline{\Omega}$, i.e., when u attains its maximum on the boundary $\partial\Omega$. Further, if u attains its positive maximum at some interior point $x_0 \in \Omega$, then all the first order partial derivatives

$$u'_{x_i}(x_0) = 0,$$

and all the second order partial derivatives

$$u''_{x_i x_i}(x_0) \leq 0,$$