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A CG-Type Method for Inverse Quadratic Eigenvalue Problems in Model Updating of Structural Dynamics

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Abstract. In this paper we first present a CG-type method for inverse eigenvalue problem of constructing real and symmetric matrices M, D and K for the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda D + K$, so that $Q(\lambda)$ has a prescribed subset of eigenvalues and eigenvectors. This method can determine the solvability of the inverse eigenvalue problem automatically. We then consider the least squares model for updating a quadratic pencil $Q(\lambda)$. More precisely, we update the model coefficient matrices M, C and K so that (i) the updated model reproduces the measured data, (ii) the symmetry of the original model is preserved, and (iii) the difference between the analytical triplet (M, D, K) and the updated triplet $(M_{\text{new}}, D_{\text{new}}, K_{\text{new}})$ is minimized. In this paper a computationally efficient method is provided for such model updating and numerical examples are given to illustrate the effectiveness of the proposed method.

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Key words: Inverse eigenvalue problem, structural dynamic model updating, quadratic pencil, iteration method.

1 Introduction

The times-invariant second order differential system

$$M\ddot{x} + D\dot{x} + Kx = f(t), \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $M, C, K \in \mathbb{R}^{n\times n}$, arises frequently in a wide scope of important appli-

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cations, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, signal processing, and finite element discretization of PDEs. It is well known that if $x(t) = ve^{\lambda t}$ represents a fundamental solution to (1.1), then the scalar λ and the vector v must solve the *quadratic eigenvalue problem* (QEP)

$$(\lambda^2 M + \lambda D + K)v = 0. \tag{1.2}$$

The scalars $\lambda \in \mathbb{C}$ and the nonzero vectors $v \in \mathbb{C}^n$ are called, respectively, eigenvalues and eigenvectors of quadratic matrix polynomial $Q(\lambda)$. Together, (λ, v) is called an eigenpair of $Q(\lambda)$. It is well known that the $Q(\lambda)$ has $2n$ finite eigenvalues over the complex field, provided the leading coefficient matrix M is nonsingular.

There are two aspects of the QEP, namely the direct problem and the inverse problem deserve attention. The direct problem analyzes and computes the spectral information, hence deducing the dynamical behavior of the system from a priori known physical parameters such as mass, elasticity, inductance and capacitance. The inverse problem determines or estimates the parameters of the system from its observed or expected eigen-information. Both problems are of significant importance in application. In this article, we consider a special inverse quadratic eigenvalue problem (IQEP) which is quite common in practice–construct the quadratic pencil with only a few eigenvalues and their corresponding eigenvectors. The IQEP that is of interest to us can be formulated as follows:

(IQEP) *(Inverse Quadratic Eigenvalue Problem)* Construct a nontrivial quadratic pencil

$$Q(\lambda) = \lambda^2 M + \lambda D + K,$$

so that its matrix coefficients (M, D, K) are of all symmetry structure and $Q(\lambda)$ has a specified set $\{(\lambda_i, \phi_i)\}_{i=1}^m$ as its eigenpairs.

Since we are only interested in real matrices, it is natural to expect that the prescribed eigenpairs are closed under complex conjugation. To facilitate the discussion, we shall described the partial eigeninformation via the pair $(\Lambda, \Phi) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times m}$ of matrices where

$$\begin{aligned}
\Lambda &= diag\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \cdots, \begin{bmatrix} \alpha_l & \beta_l \\ -\beta_l & \alpha_l \end{bmatrix}, \lambda_{2l+1}, \cdots, \lambda_m\right) \in \mathbb{R}^{m\times m}, \\
\Phi &= \left[\phi_{1R}, \phi_{1I}, \cdots, \phi_{lR}, \phi_{lI}, \phi_{2l+1}, \cdots, \phi_m\right] \in \mathbb{R}^{n\times m}.
\end{aligned}$$

Here a 2×2 block $\begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$ and the corresponding columns $[\phi_{jR}, \phi_{jI}]$ in Φ represent of store the complex conjugate pairs of eigenvalues $\alpha_j \pm i\beta_j$ and the corresponding eigenvectors $\phi_{jR} \pm \phi_{jI}$. The IQEP therefore amounts to solving the algebraic equation

$$M\Phi\Lambda^2 + D\Phi\Lambda + K\Phi = 0, \tag{1.3}$$

for the matrices M, D and K subject to symmetry structure.

By a model updating for the quadratic pencil $Q(\lambda)$, we mean to replace a portion of its original eigenstructure by some newly measured eigeninformation. It is well-known that the dynamical behavior of a vibrating system modeled by (1.2) is determined by its natural frequencies and mode shapes, that is, the eigenvalues and eigenvectors of $Q(\lambda)$. The undesired phenomenons such as instability and resonance are caused by some "troublesome" eigenvalues and the corresponding eigenvectors of $Q(\lambda)$. Therefore, in order to combat or avoid the undesired phenomenons, one way is to update the quadratic model $Q(\lambda)$ so that these "troublesome" or unfavorable eigenvalues and eigenvectors are replaced by some suitable ones. Among current developments for finite element model updating, one challenge that is of practical importance is to update the model with minimal changes. Because the solution to a MUP is not unique, therefore the notion is optimizing the adjustment or the robustness is highly plausible. Such an updating problem, which are usually faced by vibration engineers and designers, if possible, is known as the least squares model updating [1,2,4]. Moody T. Chu in [4] also pointed out that such updating problem is an area for further research. In this paper, we consider this special model updating and can be stated as follows:

(LSMUP) *(Least squares Model Updating Problem)* Given a symmetric quadratic pencil (M_0, D_0, K_0) and a few of its associated eigenpairs $(\lambda_j, \phi_j)_{j=1}^m$ with $m \le n$, assume that new eigenpairs $(\tilde{\lambda}_j, \tilde{\phi}_j)_{j=1}^m$ have been measured. Update the quadratic pencil (M_0, D_0, K_0) to a new quadratic pencil $(M_{new}, C_{new}, K_{new})$ of the same structure such that

i. the newly measured $(\varphi_j, y_j)_{j=1}^m$ form m eigenpairs of the new model $(M_{new}, C_{new}, K_{new})$;
ii. minimizing the different between the updated symmetric quadratic pencil $(M_{\text{new}}, D_{\text{new}}, K_{\text{new}})$ and the analytical symmetric quadratic pencil (M_0, D_0, K_0).

Similarly, let the real representation of the new measured eigenvalues $\{\tilde{\lambda}_j\}_{j=1}^m$ is

$$\tilde{\Lambda} = diag\left(\begin{bmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 \\ -\tilde{\beta}_1 & \tilde{\alpha}_1 \end{bmatrix}, \cdots, \begin{bmatrix} \tilde{\alpha}_l & \tilde{\beta}_l \\ -\tilde{\beta}_l & \tilde{\alpha} l \end{bmatrix}, \tilde{\lambda}_{2l+1}, \cdots, \tilde{\lambda}_m \right) \in \mathbb{R}^{m \times m}.$$

Let the real representation of the eigenvectors corresponding to the new measured eigenvalues be $\widetilde{\Phi}$.

Using the notations above, it is easy to derive that the LSMUP amounts to known the following matrix equality

$$M_0 \Phi \Lambda^2 + D_0 \Phi \Lambda + K_0 \Phi = 0,$$

we want to find a real symmetric matrix triplet $(M_{\text{new}}, D_{\text{new}}, K_{\text{new}})$ with M_{new} nonsingular satisfying the following optimization problem

$$\text{minimize} \qquad \|M_{\text{new}} - M_0\|^2 + \|D_{\text{new}} - D_0\|^2 + \|K_{\text{new}} - K_0\|^2, \tag{1.4}$$

$$\text{subject to} \qquad M_{\text{new}} \widetilde{\Phi} \tilde{\Lambda}^2 + D_{\text{new}} \widetilde{\Phi} \tilde{\Lambda} + K_{\text{new}} \widetilde{\Phi} = 0. \tag{1.5}$$

Finite element model updating has emerged in the 1990s as a significant subject to the design, construction, and maintenance of mechanical systems. This technical has widely application in damage detection and health-monitoring of the structures, such as bridges, highways, etc., and in controlling resonance vibrations in the above structures (see e.g., [22]). The application intends to correct errors in a finite element model by incorporating the measured modal data into the analytical finite element model, producing an adjusted model on the mass, damping and stiffness whose resulting behavior closely matches the experimental data. Over the years, a number of approaches has been proposed and a complete book [9] has been devoted to the subject.

The existing methods can be broadly classified into three class: (i) direct matrix model updating methods (see [3, 5–8, 10–12, 18–21]), (ii) iterative methods (see [23]) and (iii) frequency response methods. All the existing methods proposed in [11,12,20] aim at updating directly the mass, stiffness, and damping matrices in such a way that the updated model remains symmetric and reproduces the measured data as accurately as possible, but cannot guarantee that the updating with minimal changes. The method proposed in [3, 5] have the additional important feature that the eigenvalues and eigenvectors which are not updated remain unchanged by the updating procedure. This guarantees that "no spurious modes appear in the frequency range of interest". Recently a novel iterative scheme was suggested in [23] to reassign one eigenvalue at a time preserving both symmetry and no spurious in the process. The trouble is that the algorithm can break down prematurely and cannot guarantee that all desirable eigenvalues are updated.

In this paper we are concerned only with iterative matrix updating methods. We first convert the IQEP to an equivalent linear matrix equation problem, then construct a computationally efficient and symmetry preserving iterative algorithm, based on the Conjugate Gradient (CG) method, to solve the equivalence problem completely. We show that with the proposed algorithm, a desired quadratic pencil (M, D, K) of IQEP can be obtained within finitely many steps in the absence of roundoff errors. We then prove that the unique updated quadratic pencil $(M_{\rm new}, D_{\rm new}, K_{\rm new})$ of LSMUP is just the unique least norm solution of another matrix equation, which can also be obtained within finitely many steps by choosing a special kinds of initial symmetric matrix triplet. Some numerical examples are presented to show the efficiency and reliability of the proposed method for IQEP and LSMUP.

Our contribution is innovative in three areas: (i) the solvability of the IQEP can be determined automatically, which is the most significant characteristic of the proposed method, (ii) both the solution of the IQEP and the the unique solution of the LSMUP can be compute with little work and low storage requirements per iteration. In fact, it is only required to compute a residual matrix and update the iterative solution and gradient matrices *linearly* in each iteration.

The following partial notations and definitions are used throughout this paper.

- $\mathbb{R}^{n\times m}$ $-$ the set of all real $n \times m$ matrices;
- $\mathbb{S}^{n\times n}$ $-$ the set of all real $n \times m$ symmetric matrices;
- $\langle A, B\rangle = \text{trace}(B^T A) = \text{trace}(BA^T) = \sum\limits_{i=1}^{n}\sum\limits_{i=1}^{m} A_{ij}B_{ij}$; where $A, B \in \mathbb{R}^{n\times m}$;

- $\langle A, A\rangle = \|A\|^2$; the symbol$\|\cdot\|$denotes the Frobenius norm of a matrix or the Eucli -dean norm of a vector;
- $A^T =$ the transpose of A;
- $A \otimes B -$ denotes the Kronecker product of matrices A and B;
- $\mathscr{R}(M) -$ the column space of matrix M.

2 A new method for solving IQEP

To answer whether a quadratic pencil can be updated, a more fundamental question is whether a quadratic pencil can have arbitrary k prescribed eigenpairs, which is the essence of the IQEP. In order to find a quadratic pencil (M, D, K) of IQEP, we first discuss a related *matrix equation problem*, which can be described as follows:

Problem 2.1. *Given matrices $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times m}$ and $E \in \mathbb{R}^{n\times m}$. Find $X \in \mathbb{S}^{n\times n}$, $Y \in \mathbb{S}^{n\times n}$ and $Z \in \mathbb{S}^{n\times n}$ such that*

$$XA + YB + ZC = E. \tag{2.1}$$

Noting that if we denote

$$A = \Phi\Lambda^2, \quad B = \Phi\Lambda, \quad \text{and} \quad C = \Phi,$$

then find a quadratic pencil (M, D, K) of IQEP such that

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0,$$

is equivalent to find a symmetric triplet (X, Y, Z) such that (2.1) holds when $E = 0$.

We should point out, for the general cases of linear matrix equation (2.1), such as

$$\begin{aligned} &\text{(a)} \quad AXB + CYD = E, \\ &\text{(b)} \quad A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = E, \end{aligned}$$

several schemes have been proposed. For small size problems (see [16, 17]), we have the novel factorization techniques-Generalized Singular Value Decomposition (GSVD) and Canonical Correlation Decomposition (CCD); for large-scale problems, we have the gradient projection and hierarchical identification principle (see [13–15]). However, the real-life analytical model aries in vibration industries, including automobile, space and aircraft industries are generally very large, direct factorization method is not computationally feasible. Ding et al. [13–15] used the hierarchical identification principle to construct iterative solutions to the linear matrix equation (b). However, because of several serious computational difficulties, including the inversion of a possible ill-conditioned coefficient matrix and the complete loss of the exploitable structures of the unknown matrices, such as the symmetry, this approach is not practical for IQEP.

For overcoming the difficulty above-mentioned, it motivates us to construct a new iterative method to gain faster convergence and low storage requirement per iteration. The main idea is based on the classic Conjugate Gradient (CG) method. The CG method is an effective method for solving symmetric positive definite systems. The method proceeds by generating vector sequences of iterates (i.e., successive approximations to the solution), residuals corresponding to the iterates, and search directions used in updating the iterates and residuals.

The iterates $x^{(i)}$ are updated in each iteration by a multiple α_i of the search direction vector $p^{(i)}$:

$$x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}.$$

Correspondingly the residuals

$$r^{(i)} = b - Ax^{(i)},$$

are updated as

$$r^{(i)} = r^{(i-1)} + \alpha q^{(i)}, \quad \text{where } q^{(i)} = Ap^{(i)}. \tag{2.2}$$

The choice

$$\alpha = \alpha_i = \frac{{r^{(i-1)}}^T r^{(i-1)}}{{p^{(i)}}^T A p^{(i)}},$$

minimizes ${r^{(i)}}^T A^{-1} r^{(i)}$ over all possible choices for α in Eq. (2.2).

The search directions are updated using the residuals

$$p^{(i)} = r^{(i)} + \beta_{i-1} p^{(i-1)},$$

where the choice

$$\beta_i = \frac{{r^{(i)}}^T r^{(i)}}{{r^{(i-1)}}^T r^{(i-1)}},$$

ensures that $p^{(i)}$ and $Ap^{(i-1)}$-or equivalently, $r^{(i)}$ and $r^{(i-1)}$-are orthogonal. In fact, one can show that this choice of β_i makes $p^{(i)}$ and $r^{(i)}$ orthogonal to *all* previous $Ap^{(j)}$ and $r^{(j)}$ respectively.

CG method converges within at most n iterations if exact arithmetic could be performed, where n is the order of the coefficient matrix. In practice the iteration numbers may be larger than n because of the computational errors.

2.1 An extended CG method for solving Problem 2.1

In this subsection we first construct an iterative algorithm, which called the extended CG algorithm, for Problem 2.1. We then characterize its some basic properties. Finally we prove it is convergence within finitely many steps. We show that, for any arbitrary initial symmetric matrix triplet (X_0, Y_0, Z_0), a desired solution can be obtained in finitely many steps and the optimal (least norm) solution can also be obtained by choosing a special kinds of initial symmetric matrix triplet.

Extended CG Algorithm 1. *Input matrices* $A\in\mathbb{R}^{n\times m}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{n\times m}$, $E\in\mathbb{R}^{n\times m}$ *and initial matrices* $X_0\in\mathcal{S}^{n\times n}$, $Y_0\in\mathcal{S}^{n\times n}$, $Z_0\in\mathcal{S}^{n\times n}$. *Calculate*

1. $R_0 = E-(X_0A+Y_0B+Z_0C)$; $\quad Q_{0,x}=\frac{1}{2}(R_0A^T+AR_0^T)$,
   $Q_{0,y}=\frac{1}{2}(R_0B^T+BR_0^T)$, $\quad Q_{0,z}=\frac{1}{2}(R_0C^T+CR_0^T)$; $\quad k:=0$;
2. *If* $R_k=0$, *then stop; else compute*
3. $\alpha_k=\frac{\|R_k\|^2}{\|Q_{k,x}\|^2+\|Q_{k,y}\|^2+\|Q_{k,z}\|^2}$; $\quad X_{k+1}=X_k+\alpha_kQ_{k,x}$,
   $Y_{k+1}=X_k+\alpha_kQ_{k,y}$, $\quad Z_{k+1}=X_k+\alpha_kQ_{k,z}$;
   $R_{k+1}=R_k-\alpha_k(Q_{k,x}A+Q_{k,y}B+Q_{k,z}C)$;
   $\beta_k=\frac{\|R_{k+1}\|^2}{\|R_k\|^2}$; $\quad Q_{k+1,x}=\frac{1}{2}(R_{k+1}A^T+AR_{k+1}^T)+\beta_kQ_{k,x}$,
   $Q_{k+1,y}=\frac{1}{2}(R_{k+1}B^T+BR_{k+1}^T)+\beta_kQ_{k,y}$, $\quad Q_{k+1,z}=\frac{1}{2}(R_{k+1}C^T+CR_{k+1}^T)+\beta_kQ_{k,z}$;
4. *Let* $k:=k+1$ *and go to Step 2.*

Remark 2.1. Since $Q_{0,x}\in\mathbb{S}^{n\times n}$ by assumption and $R_{k+1}A^T+AR_{k+1}^T\in\mathbb{S}^{n\times n}$ for all k, the third equation from button in Step 3 and induction imply that $Q_{k+1,x}\in\mathbb{S}^{n\times n}$. Since $X_0\in\mathbb{S}^{n\times n}$ by assumption and $Q_{k,x}\in\mathbb{S}^{n\times n}$ for all k, the second equation in Step 3 and induction imply that $X_{k+1}\in\mathbb{S}^{n\times n}$. Analogously, we have $Q_{k,y}\in\mathbb{S}^{n\times n}$, $Q_{k,z}\in\mathbb{S}^{n\times n}$, $Y_k\in\mathbb{S}^{n\times n}$, and $Z_k\in\mathbb{S}^{n\times n}$, for all k. R_k is the residual of Eq. (2.1), where $k=0,1,2,\cdots$.

Lemma 2.1. *Suppose Eq.* (2.1) *is consistent over symmetric matrix triplet, and* $[X_*,Y_*,C_*]$ *is an arbitrary solution. Then for any initial symmetric matrix triplet* $[X_0,Y_0,Z_0]$, *the sequences* X_i, Y_i, Z_i R_i, $Q_{i,x}$, $Q_{i,y}$ *and* $Q_{i,z}$ *generated by Algorithm 1 satisfy*

$$\langle Q_{i,x},\ X_*-X_i\rangle+\langle Q_{i,y},Y_*-Y_i\rangle+\langle Q_{i,z},Z_*-Z_i\rangle=\|R_i\|^2,\quad i=0,1,2,\cdots.$$

Proof. We prove the conclusion by induction. When $i=0$, we have

$$\begin{aligned}
&\langle Q_{0,x},\ X_*-X_0\rangle+\langle Q_{0,y},\ Y_*-Y_0\rangle+\langle Q_{0,z},Z_*-Z_0\rangle\\
=&\langle\frac{1}{2}(R_0A^T+AR_0^T),\ X_*-X_0\rangle+\langle\frac{1}{2}(R_0B^T+BR_0^T),\ Y_*-Y_0\rangle\\
&+\langle\frac{1}{2}(R_0C^T+CR_0^T),\ Z_*-Z_0\rangle\\
=&\langle R_0,\ (X_*-X_0)A\rangle+\langle R_0,\ (Y_*-Y_0)B\rangle+\langle R_0,\ (Z_*-Z_0)A\rangle\\
=&\langle R_0,\ X_*A+Y_*B+Z_*C-(X_0A+Y_0B+Z_0C)\rangle\\
=&\langle R_0,\ E-(X_0A+Y_0B+Z_0C)\rangle\\
=&\|R_0\|^2.
\end{aligned}$$

Suppose that the conclusion holds for $i=v, v\geq 0$, that is,

$$\langle Q_{v,x},\ X_*-X_v\rangle+\langle Q_{v,y},Y_*-Y_v\rangle+\langle Q_{v,z},\ Z_*-Z_v\rangle=\|R_v\|^2.$$

Then, when $i = v+1$, we have

$$\begin{aligned}
&\langle Q_{v+1,x},\ X_* - X_{v+1}\rangle + \langle Q_{v+1,y},\ Y_* - Y_{v+1}\rangle + \langle Q_{v+1,z},\ Z_* - Z_{v+1}\rangle \\
=&\langle \frac{1}{2}(R_{v+1}A^T + AR_{v+1}^T) - \beta_v Q_{v,x},\ X_* - X_{v+1}\rangle + \langle \frac{1}{2}(R_{v+1}B^T + BR_{v+1}^T) \\
&\quad - \beta_v Q_{v,y},\ Y_* - Y_{v+1}\rangle + \langle \frac{1}{2}(R_{v+1}C^T + CR_{v+1}^T) - \beta_v Q_{v,z},\ Z_* - Z_{v+1}\rangle \\
=&\langle R_{v+1},\ X_*A + Y_*B + Z_*C - (X_{v+1}A + Y_{v+1}B + Z_{v+1}C)\rangle \\
&\quad + \beta_v\big\{\langle Q_{v,x},\ X_* - X_{v+1}\rangle + \langle Q_{v,y},\ Y_* - Y_{v+1}\rangle + \langle Q_{v,z},\ Z_* - Z_{v+1}\rangle\big\} \\
=&\langle R_{v+1},\ E - (X_{v+1}A + Y_{v+1}B + Z_{v+1}C)\rangle \\
&\quad + \beta_v\big\{\langle Q_{v,x},\ X_* - (X_v + \alpha_v Q_{v,x})\rangle + \langle Q_{v,y},\ Y_* - (Y_v + \alpha_v Q_{v,y})\rangle \\
&\quad + \langle Q_{v,z},\ Z_* - (Z_v + \alpha_v Q_{v,z})\rangle\big\} \\
=&\|R_{v+1}\|^2 + \beta_v\big\{\langle Q_{v,x},\ X_* - X_v\rangle + \langle Q_{v,y},\ Y_* - Y_v\rangle + \langle Q_{v,z},\ Z_* - Z_v\rangle \\
&\quad - \alpha_v(\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2 + \|Q_{v,z}\|^2)\big\} \\
=&\|R_{v+1}\|^2 - \beta_v(\|R_v\|^2 - \|R_v\|^2) = \|R_{v+1}\|^2.
\end{aligned}$$

By the principle of induction, the desired conclusion is obtained $\square$

Remark 2.2. From the formula of $Q_{v,x}$, $Q_{v,y}$ and $Q_{v,z}$ in Algorithm 1 and Lemma 2.1, we know that if Eq. (2.1) is consistent, then, $R_i = 0$ if and only if

$$Q_{v,x} = 0, \quad Q_{v,y} = 0, \quad \text{and} \quad Q_{v,z} = 0.$$

In other words, $\|R_i\| \neq 0$ if and only if

$$\|Q_{v,x}\|^2 + \|Q_{v,y}\|^2 + \|Q_{v,z}\|^2 \neq 0.$$

This results implies that if there exists a positive number k such that

$$Q_{v,x} = 0, \quad Q_{v,y} = 0, \quad \text{and} \quad Q_{v,z} = 0, \quad \text{but} \quad R_k \neq 0,$$

then Eq. (2.1) is inconsistent. Hence, the solvability of Eq. (2.1) can be determined automatically by Algorithm 1 in the absence of round-off errors.

Lemma 2.2. *Assume that Eq.* (2.1) *is consistent and the sequences R_i, $Q_{v,x}$, $Q_{v,y}$ and $Q_{v,z}$, where $R_i \neq 0, i = 0,1,2,\cdots,k$, are generated by Algorithm 1. Then*

$$\begin{aligned}
&\langle R_i, R_j\rangle = 0, \\
&\langle Q_{i,x}, Q_{j,x}\rangle + \langle Q_{i,y}, Q_{j,y}\rangle + \langle Q_{i,z}, Q_{j,z}\rangle = 0, \quad i,j = 0,1,2,\cdots,k, \ \ i \neq j,
\end{aligned}$$

Proof. Since $\langle A, B\rangle = \langle B, A\rangle$ for all matrices A and B in $\mathbb{R}^{n\times m}$, we only prove the conclusion hold for all $0 \le i < j \le k$. To this end, using induction and two steps are required.

1. Show that

$$\langle R_i, R_{i+1}\rangle = 0, \quad \text{and} \quad \langle Q_{i,x}, Q_{i+1,x}\rangle + \langle Q_{i,y}, Q_{i+1,y}\rangle + \langle Q_{i,z}, Q_{i+1,z}\rangle = 0,$$

for all $i = 0, 1, 2, \cdots, k$. To prove this conclusion, we also use induction. For $i = 0$, we have

$$\begin{aligned}
\langle R_0, R_1\rangle =& \langle R_0, R_0 - \alpha_0(Q_{0,x}A + Q_{0,y}B + Q_{0,z}C)\rangle \\
=& \|R_0\|^2 - \alpha_0\big(\langle R_0A^T, Q_{0,x}\rangle + \langle R_0B^T, Q_{0,y}\rangle + \langle R_0C^T, Q_{0,z}\rangle\big) \\
=& \|R_0\|^2 - \alpha_0\Big(\langle \frac{1}{2}(R_0A^T + AR_0^T), Q_{0,x}\rangle + \langle \frac{1}{2}(R_0B^T + BR_0^T), Q_{0,y}\rangle \\
& + \langle \frac{1}{2}(R_0C^T + CR_0^T), Q_{0,z}\rangle\Big) \\
=& \|R_0\|^2 - \alpha_0\Big(\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2 + \|Q_{0,y}\|^2\Big) = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \langle Q_{0,x}, Q_{1,x}\rangle + \langle Q_{0,y}, Q_{1,y}\rangle + \langle Q_{0,z}, Q_{1,z}\rangle \\
=& \langle Q_{0,x}, \frac{1}{2}(R_1A^T + AR_1^T) - \beta_0 Q_{0,x}\rangle + \langle Q_{0,y}, \frac{1}{2}(R_1B^T + BR_1^T) - \beta_0 Q_{0,y}\rangle \\
& + \langle Q_{0,z}, \frac{1}{2}(R_1C^T + CR_1^T) - \beta_0 Q_{0,z}\rangle \\
=& \langle Q_{0,x}A + Q_{0,y}B + Q_{0,z}C, R_1\rangle - \beta_0(\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2 + \|Q_{0,y}\|^2) \\
=& \frac{1}{\alpha_0}\langle R_0 - R_1, R_1\rangle - \beta_0(\|Q_{0,x}\|^2 + \|Q_{0,y}\|^2 + \|Q_{0,y}\|^2) = 0.
\end{aligned}$$

Assume that the conclusion holds for all $i \leq s,\ 0 < s < k$. Then

$$\begin{aligned}
\langle R_s, R_{s+1}\rangle =& \langle R_s, R_s - \alpha_s(Q_{s,x}A + Q_{s,y}B + Q_{s,z}C)\rangle \\
=& \|R_s\|^2 - \alpha_s\big(\langle R_sA^T, Q_{s,x}\rangle + \langle R_sB^T, Q_{s,y}\rangle + \langle R_sC^T, Q_{s,z}\rangle\big) \\
=& \|R_s\|^2 - \alpha_s\Big(\langle \frac{1}{2}(R_sA^T + AR_s^T), Q_{s,x}\rangle + \langle \frac{1}{2}(R_sB^T + BR_s^T), Q_{s,y}\rangle \\
& + \langle \frac{1}{2}(R_sC^T + CR_s^T), Q_{s,z}\rangle\Big) \\
=& \|R_s\|^2 - \alpha_s\Big(\langle Q_{s,x} - \beta_{s-1}Q_{s-1,x}, Q_{s,x}\rangle + \langle Q_{s,y} - \beta_{s-1}Q_{s-1,y}, Q_{s,y}\rangle \\
& + \langle Q_{s,z} - \beta_{s-1}Q_{s-1,z}, Q_{s,z}\rangle\Big) \\
=& \|R_s\|^2 - \alpha_s\Big(\|Q_{s,x}\|^2 + \|Q_{s,y}\|^2 + \|Q_{s,y}\|^2\Big) = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \langle Q_{s,x}, Q_{s+1,x}\rangle + \langle Q_{s,y}, Q_{s+1,y}\rangle + \langle Q_{s,z}, Q_{s+1,z}\rangle \\
=& \langle Q_{s,x}, \frac{1}{2}(R_{s+1}A^T + AR_{s+1}^T) + \beta_s Q_{s,x}\rangle + \langle Q_{s,y}, \frac{1}{2}(R_{s+1}B^T + BR_{s+1}^T) + \beta_s Q_{s,y}\rangle \\
& + \langle Q_{s,z}, \frac{1}{2}(R_{s+1}C^T + CR_{s+1}^T) + \beta_s Q_{s,z}\rangle \\
=& \langle Q_{s,x}A + Q_{s,y}B + Q_{s,z}C, R_{s+1}\rangle + \beta_s\Big(\|Q_{s,x}\|^2 + \|Q_{s,y}\|^2 + \|Q_{s,y}\|^2\Big)
\end{aligned}$$

$$=\frac{1}{\alpha_s}\langle R_s - R_{s+1}, R_{s+1}\rangle + \beta_s\Big(\|Q_{s,x}\|^2 + \|Q_{s,y}\|^2 + \|Q_{s,y}\|^2\Big) = 0.$$

By the principle of induction,

$$\langle R_i, R_{i+1}\rangle = 0, \quad \text{and} \quad \langle Q_{i,x}, Q_{i+1,x}\rangle + \langle Q_{i,y}, Q_{i+1,y}\rangle + \langle Q_{i,z}, Q_{i+1,z}\rangle = 0,$$

hold true for all $i = 0, 1, 2, \cdots, k$.

2. Assume that

$$\langle R_i, R_{i+l}\rangle = 0, \quad \text{and} \quad \langle Q_{i,x}, Q_{i+l,x}\rangle + \langle Q_{i,y}, Q_{i+l,y}\rangle + \langle Q_{i,z}, Q_{i+l,z}\rangle = 0,$$

for all $0 \le i \le k$ and $1 < l < k$, we will show that

$$\langle R_i, R_{i+l+1}\rangle = 0, \quad \text{and} \quad \langle Q_{i,x}, Q_{i+l+1,x}\rangle + \langle Q_{i,y}, Q_{i+l+1,y}\rangle + \langle Q_{i,z}, Q_{i+l+1,z}\rangle = 0.$$

The proof is as follows:

$$\begin{aligned}
\langle R_i, R_{i+l+1}\rangle =& \langle R_i, R_{i+l} - \alpha_{i+l}(Q_{i+l,x}A + Q_{i+l,y}B + Q_{i+l,z}C)\rangle \\
=& -\alpha_{i+l}\big(\langle R_iA^T, Q_{i+l,x}\rangle + \langle R_iB^T, Q_{i+l,y}\rangle + \langle R_iC^T, Q_{i+l,z}\rangle\big) \\
=& -\alpha_{i+l}\Big(\langle\frac{1}{2}(R_iA^T + AR_i^T), Q_{i+l,x}\rangle + \langle\frac{1}{2}(R_iB^T + BR_i^T), Q_{i+l,y}\rangle \\
&+ \langle\frac{1}{2}(R_iC^T + CR_i^T), Q_{i+l,z}\rangle\Big) \\
=& -\alpha_{i+l}\big(\langle Q_{i,x} - \beta_{i-1}Q_{i-1,x}, Q_{i+l,x}\rangle + \langle Q_{i,y} - \beta_{i-1}Q_{i-1,y}, Q_{i+l,y}\rangle \\
&+ \langle Q_{i,z} - \beta_{i-1}Q_{i-1,z}, Q_{i+l,z}\rangle\big) \\
=& -\alpha_{i+l}\big\{\langle Q_{i,x}, Q_{i+l,x}\rangle + \langle Q_{i,y}, Q_{i+l,y}\rangle + \langle Q_{i,z}, Q_{i+l,z}\rangle \\
&- \beta_{i-1}(\langle Q_{i-1,x}, Q_{i+l,x}\rangle + \langle Q_{i-1,y}, Q_{i+l,y}\rangle + \langle Q_{i-1,z}, Q_{i+l,z}\rangle)\big\} \\
=&0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle Q_{i,x}, Q_{i+l+1,x}\rangle + \langle Q_{i,y}, Q_{i+l+1,y}\rangle + \langle Q_{i,z}, Q_{i+l+1,z}\rangle \\
=&\langle Q_{i,x}, \frac{1}{2}(R_{i+l+1}A^T + AR_{i+l+1}^T) + \beta_{i+l}Q_{i+l,x}\rangle \\
&+ \langle Q_{i,y}, \frac{1}{2}(R_{i+l+1}B^T + BR_{i+l+1}^T) + \beta_{i+l}Q_{i+l,y}\rangle \\
&+ \langle Q_{i,z}, \frac{1}{2}(R_{i+l+1}C^T + CR_{i+l+1}^T) + \beta_{i+l}Q_{i+l,z}\rangle \\
=&\langle Q_{i,x}A + Q_{i,y}B + Q_{i,z}C, R_{i+l+1}\rangle + \beta_{i+l}(\langle Q_{i,x}, Q_{i+l,x}\rangle \\
&+ \langle Q_{i,y}, Q_{i+l,y}\rangle + \langle Q_{i,z}, Q_{i+l,z}\rangle) \\
=&\frac{1}{\alpha_i}\langle R_i - R_{i+1}, R_{i+l+1}\rangle = 0.
\end{aligned}$$

From Steps 1 and 2, we have by induction that

$$\langle R_i, R_j\rangle = 0, \quad \text{and} \quad \langle Q_{i,x}, Q_{j,x}\rangle + \langle Q_{i,y}, Q_{j,y}\rangle + \langle Q_{i,z}, Q_{j,z}\rangle = 0,$$

hold for all $i, j = 0, 1, 2, \cdots, k,\ \ i \neq j$. □

Remark 2.3. Lemma 2.2 implies that if Eq. (2.1) is consistent, then, for any initial symmetric matrix triplet (X_0, Y_0, Z_0), a solution can be obtained within at most nm iteration steps. Since R_0, R_1, $R_2, \cdots$, are orthogonal each other in a finite dimension matrix space $R^{n\times m}$, it is certain that there exists a positive number $k \leq nm$ such that $R_k = 0$.

To facilitate the statement of our main results, we introduce the following lemma.

Lemma 2.3. *Suppose that the consistent system of linear equations $Mx = b$ has a solution $x \in \mathscr{R}(M^T)$. Then x^* is an unique least Frobenius norm solution of the system of linear equations.*

Proof. We decompose the matrix $M \in R^{m\times n}$ by singular value decomposition (SVD):

$$M = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T,$$

where

$$\begin{aligned} &U = (U_1, U_2) \in \mathbb{OR}^{m\times m}, && V = (V_1, V_2) \in \mathbb{OR}^{n\times n}, && U_1 \in \mathbb{R}^{m\times r}, \\ &\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_r) > 0, && r = rank(M), && V_1 \in \mathbb{R}^{n\times r}. \end{aligned}$$

Then the Moore-Penrose generalized inverse of the matrix M is

$$M^+ = V_1 \Sigma^{-1} U_1^T,$$

and the general solution of the system of linear equation $Mx = b$ is

$$x = M^+ b + (I - M^+ M) z,$$

where z is an arbitrary vector of suitable dimension. Since

$$M^+ = V_1 \Sigma^{-1} U_1^T \in \mathscr{R}(V_1), \qquad (I - M^+ M) z = V_2 V_2^T \in \mathscr{R}(V_2),$$

V_1 and V_2 are orthogonal to each other, then $M^+ b$ is the unique least Frobenius norm solution of the system of linear equations $Mx = b$. On the other hand, since

$$M^T = V_1 \Sigma U_1^T,$$

and the solution $x^* \in \mathscr{R}(M^T)$, then $x^* \in R(V_1)$. Therefore, x^* is the least Frobenius norm solution of the system of linear equations $Mx = b$, that is,

$$x^* = M^+ b,$$

so the proof is complete. $\square$

For any matrix $M \in R^{m\times n}$, denoted by $vec(A)$, the following mn-vector containing all the entries of matrix M:

$$vec(M) = \begin{pmatrix} M(:,1) \\ M(:,2) \\ \vdots \\ M(:,n) \end{pmatrix} \in \mathbb{R}^{mn},$$

where $M(:,i)$ denotes ith column of matrix M (i.e., Matlab style). For vector $\chi \in \mathbb{R}^{mn}$, denoted by $\widetilde{vec}_{m,n}(\chi)$ the following $m\times n$ matrix containing all the entries of vector χ:

$$\widetilde{vec}_{m,n}(\chi) = \big(\chi(1:m)\ \chi(m+1:2m)\ \cdots\ \chi[(n-1)m+1:mn]\big) \in \mathbb{R}^{m\times n},$$

where $\chi(i:j)$ denotes a vector containing the elements i to j of vector χ.

We know the solvability of Eq. (2.1) over symmetric matrix triple is equivalent to the following linear system

$$\begin{cases} XA+YB+ZC=E, \\ A^TX+B^TY+C^TZ=E^T. \end{cases}$$

Then the above system of matrix equations is equivalent to the systems of linear equations

$$\begin{pmatrix} A^T\otimes I_n & B^T\otimes I_n & C^T\otimes I_n \\ I_n\otimes A & I_n\otimes B & I_n\otimes C \end{pmatrix} \begin{pmatrix} vec(X) \\ vec(Y) \\ vec(Z) \end{pmatrix} = \begin{pmatrix} vec(E) \\ vec(E^T) \end{pmatrix}.$$

Note that

$$\begin{aligned}
\begin{pmatrix} vec(HA^T+AH^T) \\ vec(HB^T+BH^T) \\ vec(HC^T+CH^T) \end{pmatrix} &= \begin{pmatrix} A\otimes I_n & I_n\otimes A^T \\ B\otimes I_n & I_n\otimes B^T \\ C\otimes I_n & I_n\otimes C^T \end{pmatrix} \begin{pmatrix} vec(H) \\ vec(H) \\ vec(H) \end{pmatrix} \\
&= \begin{pmatrix} A^T\otimes I_n & B^T\otimes I_n & C^T\otimes I_n \\ I_n\otimes A & I_n\otimes B & I_n\otimes C \end{pmatrix}^T \begin{pmatrix} vec(H) \\ vec(H) \\ vec(H) \end{pmatrix} \\
&\in \mathscr{R}\left(\begin{pmatrix} A^T\otimes I_n & B^T\otimes I_n & C^T\otimes I_n \\ I_n\otimes A & I_n\otimes B & I_n\otimes C \end{pmatrix}^T \right).
\end{aligned}$$

We see that if we let initial matrices

$$X_0 = HA^T+AH^T, \quad Y_0 = HB^T+BH^T, \quad \text{and} \quad Z_0 = HC^T+CH^T,$$

where $H \in \mathbb{R}^{n\times m}$ is arbitrary, then all X_k, Y_k and Z_k, generated by Algorithm 1, satisfy

$$\begin{pmatrix} vec(X_k) \\ vec(Y_k) \\ vec(Z_k) \end{pmatrix} \in \mathscr{R}\left(\begin{pmatrix} A^T\otimes I_n & B^T\otimes I_n & C^T\otimes I_n \\ I_n\otimes A & I_n\otimes B & I_n\otimes C \end{pmatrix}^T \right).$$

Hence, from Lemma 2.3 we obtain that if (X^*, Y^*, Z^*), generated by Algorithm 1, is the solution triplet of the matrix equation (2.1), then it is its least Frobenius norm solution pair. In this case, X^*, Y^* and Z^* can be expressed as

$$X^* = \widetilde{vec}_{n,n}[W_*(1 : n^2)], \tag{2.3a}$$
$$Y^* = \widetilde{vec}_{n,n}[W_*(n^2+1 : 2n^2)], \tag{2.3b}$$
$$Z^* = \widetilde{vec}_{n,n}[W_*(2n^2+1, 3n^2)], \tag{2.3c}$$

where

$$W_* = \begin{pmatrix} A^T \otimes I_n & B^T \otimes I_n & C^T \otimes I_n \\ I_n \otimes A & I_n \otimes B & I_n \otimes C \end{pmatrix}^+ \begin{pmatrix} vec(E) \\ vec(E^T) \end{pmatrix}.$$

The above conclusions on the solution of the linear matrix equation (2.1) can be collected in the following theorem. Its proof is omitted.

Theorem 2.1. *Assume that Eq.* (2.1) *is consistent. Then for any initial guess symmetric matrix triplet (X_0, Y_0, Z_0), the matrix triplet sequence (X_k, Y_k, Z_k), generated by Algorithm 1, converges to its solution within at most n^2 iteration steps. Furthermore, if we choose the initial guess matrices*

$$X_0 = HA^T + AH^T, \quad Y_0 = HB^T + BH^T, \quad \textit{and} \quad Z_0 = HC^T + CH^T,$$

with H is arbitrary, or more specifically, if we let

$$X_0 = 0, \quad Y_0 = 0, \quad \textit{and} \quad Z_0 = 0,$$

then the solution triplet $[X^*, Y^*, Z^*]$ obtained by Algorithm 1 is the least Frobenius norm solution of the matrix equation (2.1). *In this case, X^*, Y^* and Z^* can be expressed as* (2.3).

Remark 2.4. If $E = 0$, we know that the homogeneous linear matrix equation

$$XA + YB + CZ = 0,$$

is always solvable. In this case we say the IQEP is unsolvable if it has an unique null solution. Since

$$XA + YB + CZ = 0 \Leftrightarrow \begin{pmatrix} A^T \otimes I_n & B^T \otimes & I_n \, C^T \otimes I_n \end{pmatrix} \begin{pmatrix} vec(X) \\ vec(Y) \\ vec(Y) \end{pmatrix} = 0.$$

Therefore, if

$$rank\big(\begin{pmatrix} A^T \otimes I_n & B^T \otimes & I_n \, C^T \otimes I_n \end{pmatrix} \big) = 3n,$$

the IQEP is unsolvable.

2.2 Numerical experiments for IQEP

In this subsection we illustrate the feasibility and efficiency of the proposed algorithm by using some numerical examples. All numerical implementations were performed on a personal computer of the Intel P4 2.4 GHz processor family with 512M memory using Matlab 7.0.

Example 2.1. Consider the IQEP where the partial eigenstructure $(\Lambda,\Phi)\in\mathbb{R}^{5\times5}\times\mathbb{R}^{5\times5}$ is randomly. Assume

$$\Lambda=\begin{bmatrix}-0.2168 & -4.3159 & 0 & 0 & 0\\ 4.3159 & -0.2168 & 0 & 0 & 0\\ 0 & 0 & 2.0675 & -0.9597 & 0\\ 0 & 0 & 0.9597 & 2.0675 & 0\\ 0 & 0 & 0 & 0 & -0.3064\end{bmatrix},$$

$$\Phi=\begin{bmatrix}-0.4132 & 5.2801 & 2.9437 & -6.6098 & -9.6715\\ -4.3518 & 3.2758 & -5.1656 & 9.1024 & -9.1357\\ -0.1336 & -4.0588 & 2.5321 & 3.3049 & -4.4715\\ -5.1414 & 4.4003 & -2.2721 & 5.2872 & 6.9659\\ 8.6146 & -4.0112 & -6.9380 & 1.4345 & -4.4708\end{bmatrix}.$$

We first let $A=\Phi\Lambda^2$, $B=\Phi\Lambda$, $C=\Phi$ and $E=0\in\mathbb{R}^{5\times5}$, and the initial matrix $X_0=I_5$, $Y_0=I_5$ and $Z_0=I_5$. Using Algorithm 1 and iterate 42 steps, we have the quadratic pencil (M,D,K) of IQEP as follows:

$$M=\begin{bmatrix}0.3635 & 0.0841 & 0.2671 & -0.0256 & 0.0943\\ 0.0841 & 0.3427 & -0.0391 & -0.4227 & -0.0642\\ 0.2671 & -0.0391 & 0.3843 & 0.0722 & 0.1013\\ -0.0256 & -0.4227 & 0.0722 & 0.6217 & 0.1567\\ 0.0943 & -0.0642 & 0.1013 & 0.1567 & 0.1268\end{bmatrix},$$

$$C=\begin{bmatrix}0.5696 & 0.4942 & 0.4271 & 0.1148 & 0.2090\\ 0.4942 & 0.4286 & 0.1481 & -0.4788 & -0.0298\\ 0.4271 & 0.1481 & 0.1903 & 0.1508 & 0.2615\\ 0.1148 & -0.4788 & 0.1508 & 0.6867 & 0.1426\\ 0.2090 & -0.0298 & 0.2615 & 0.1426 & -0.0456\end{bmatrix},$$

$$K=\begin{bmatrix}0.6354 & 0.0541 & -0.1566 & 0.3655 & -0.0596\\ 0.0541 & 0.4321 & -0.0742 & -0.0519 & -0.2808\\ -0.1566 & -0.0742 & 0.7288 & 0.0898 & 0.2619\\ 0.3655 & -0.0519 & 0.0898 & 0.7057 & 0.0192\\ -0.0596 & -0.2808 & 0.2619 & 0.0192 & 0.5835\end{bmatrix}.$$

In Fig. 1 we characterize the convergence curve for the Frobenius norm of the residual and the iterative variable $(\|Q_{k,x}\|^2+\|Q_{k,y}\|^2+\|Q_{k,y}\|^2)^{1/2}$.

Table 1: The residual for Example 2.1.

(λ_i,ϕ_i)	(λ_1,ϕ_1)	(λ_2,ϕ_2)	(λ_3,ϕ_3)	(λ_4,ϕ_4)	(λ_5,ϕ_5)
res(λ_i,ϕ_i)	4.8122e-011	4.8122e-011	1.3726e-011	1.3726e-011	1.1801e-013

It is easy to compute

$$\|M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi\|^2 = 8.3630 * 10^{-22}.$$

We define the residual as

$$res(\lambda_i,\phi_i) = \|(\lambda_i^2 M + \lambda_i D + K)\phi_i\|,$$

and the numerical results shown in Table 1.

Therefore, the prescribed eigenvalues (the diagonal elements of the matrix Λ) and eigenvectors (the column vectors of the matrix Φ) are embedded in the model

$$(\lambda^2 M + \lambda C + K)v = 0.$$

Example 2.2. In this experiment, the prescribed eigenvalue matrix Λ and eigenvector matrix Φ are randomly constructed as follows:

$$T = rand(8),\quad G = rand(r);\quad \Lambda = diag(T);\quad [P,L] = eig(G);\quad \Phi = P(:,1:8);$$

where r are a constant that determine the magnitudes of the quadratic pencil $Q(\lambda)$. In Table 2, we list our numerical results as the matrix size is variant from $r = 100$ to $r = 400$. We set the initial iterative matrices be $X_0 = I_r$, $Y_0 = I_r$, $Z_0 = I_r$. Then a symmetric matrix triplet (M,D,K) can be obtained by using Algorithm 1. In this table, we list CPU times(s), iteration numbers (k), residual norm $\|R_k\|^2$ and the iterative

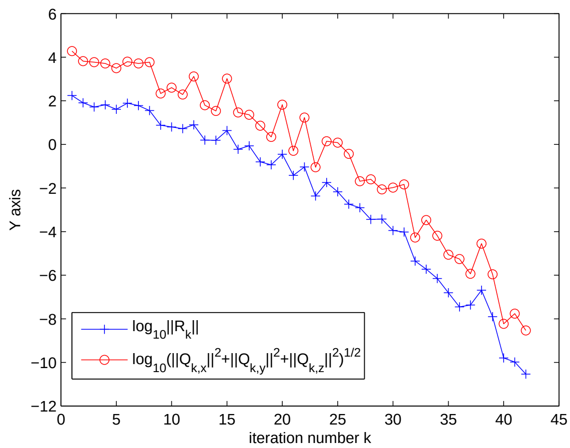

Figure 1: Convergence curve for the Frobenius norm of the residual and the iterative variable $(\|Q_{k,x}\|^2 + \|Q_{k,y}\|^2 + \|Q_{k,y}\|^2)^{1/2}$.

Table 2: CPU times, etc for Example 2.2.

	r=100	r=200	r=300	r=400
iteration number k	407	484	340	344
elapsed times(s)	20.596916	96.949634	167.819220	320.679199
residual $\|R_k\|^2$	9.9635e-021	9.7405e-021	4.6034e-021	2.1334e-021
$\|Q_{k,x}\|^2+\|Q_{k,y}\|^2+\|Q_{k,y}\|^2$	7.1043e-021	4.7163e-021	2.1680e-021	1.5487e-021

Table 3: The residual for Example 2.2.

	r=100	r=200	r=300	r=400
eigenpairs	$\|Q_1(\lambda_j)\phi_j\|$	$\|Q_2(\lambda_j)\phi_j\|$	$\|Q_3(\lambda_j)\phi_j\|$	$\|Q_4(\lambda_j)\phi_j\|$
(λ_1,ϕ_1)	3.3096e-011	3.0144e-011	2.1793e-011	1.5976e-011
(λ_2,ϕ_2)	3.8608e-011	2.5528e-011	2.3769e-011	2.8385e-011
(λ_3,ϕ_3)	2.6605e-011	1.3448e-011	3.1068e-011	1.4543e-011
(λ_4,ϕ_4)	3.4524e-011	2.9526e-011	2.9004e-011	1.2856e-011
(λ_5,ϕ_5)	3.8570e-011	3.6197e-011	2.2360e-011	5.2986e-012
(λ_6,ϕ_6)	3.0067e-011	2.6161e-011	2.0759e-011	1.2493e-011
(λ_7,ϕ_7)	4.1647e-011	5.1522e-011	1.9335e-011	1.5576e-011
(λ_8,ϕ_8)	3.6767e-011	4.9783e-011	2.1266e-011	1.6400e-011

variable $(\|Q_{k,x}\|^2+\|Q_{k,y}\|^2+\|Q_{k,y}\|^2)^{1/2}$ for different values of r with the stopping criteria $\|R_k\|\le 10^{-10}$.

In Table 3, we show the residual $\|Q(\lambda_i)\phi_i\|$ for different values of r, where (λ_i,ϕ_i) are the computed eigenpairs of $Q(\lambda)$, for $j=1,\cdots,8$.

3 The extended CG algorithm for solving LSMUP

3.1 Numerical method for LSMUP

Recall that a least squares model updating with no spill-over is mathematically equivalent to a optimal approximation problem and a IQEP with prescribed eigenvalue $\tilde{\Lambda}$ and eigenvector $\tilde{\Phi}$, with the pair $(\tilde{\Lambda},\tilde{\Phi})\in\mathbb{R}^{m\times m}\times\mathbb{R}^{n\times m}$ representing the portion of eigenstructure that has been modified.

Denote $\widetilde{\Phi}\tilde{\Lambda}^2=\tilde{A}$, $\widetilde{\Phi}\tilde{\Lambda}=\tilde{B}$, and $\widetilde{\Phi}=\tilde{C}$. Then Eq. (1.5) is equivalent to the algebraic equation

$$
\begin{aligned}
&(M_{\rm new}-M_0)\tilde{A}+(D_{\rm new}-D_0)\tilde{B}+(K_{\rm new}-K_0)\tilde{C} &(3.1)\\
=&-(M_0A+D_0B+K_0C). &(3.2)
\end{aligned}
$$

We also let

$$
\begin{aligned}
&M_{\rm new}-M_0=X, && D_{\rm new}-D_0=Y,\\
&K_{\rm new}-K_0=Z, && \tilde{E}=-(M_0A+D_0B+K_0C).
\end{aligned}
$$

Then finding the updated symmetric matrix triplet $(M_{\text{new}}, D_{\text{new}}, K_{\text{new}})$ for the optimization problem (1.4)-(1.5) is equivalent to finding the unique symmetric matrix triplet (X, Y, Z) for the following new optimization problem

$$\text{minimize} \;\; \|X\|^2 + \|Y\|^2 + \|Z\|^2, \tag{3.3}$$

$$\text{subject to} \;\; X\tilde{A} + Y\tilde{B} + Z\tilde{C} = \tilde{E}. \tag{3.4}$$

By using Algorithm 1 and Theorem 2.1, and letting the initial matrices

$$X_0 = 0 \in \mathbb{R}^{n\times n}, \quad Y_0 = 0 \in \mathbb{R}^{n\times n}, \quad \text{and} \quad Z_0 = 0 \in \mathbb{R}^{n\times n},$$

we can obtain the unique least norm symmetric solution triplet (X^*, Y^*, Z^*) of Eq. (3.4). Once (X^*, Y^*, Z^*) is known, the unique updated symmetric solution triplet $(M_{\text{new}}, D_{\text{new}}, K_{\text{new}})$ of the optimization problem (1.4)-(1.5) can be computed. In this case,

$$M_{\text{new}} = X^* + M_0, \quad D_{\text{new}} = Y^* + D_0, \quad \text{and} \quad K_{\text{new}} = Z^* + K_0.$$

3.2 Numerical experiments for LSMUP

Example 3.1. A mass-spring system of 10 DoF.

Consider the example of a mass-spring system of 10 DoF, as depicted in Fig. 2. In this example all rigid bodies have a mass of 1kg,and all springs have stiffness 1 kN/m. The analytical model is given by

$$M_0 = eye(10),$$

$$D_0 = \begin{pmatrix} 0.4810 & -8.3809 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8.3809 & 8.3809 & -1.0254 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.0254 & 1.0254 & -7.2827 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7.2827 & 7.2827 & -4.4050 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.4050 & 4.4050 & -9.9719 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9.9719 & 9.9719 & -5.6247 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5.6247 & 5.6247 & -4.6585 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4.6585 & 4.6585 & -4.1901 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4.1901 & 4.1901 & -2.1160 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.1160 & 2.1160 \end{pmatrix},$$

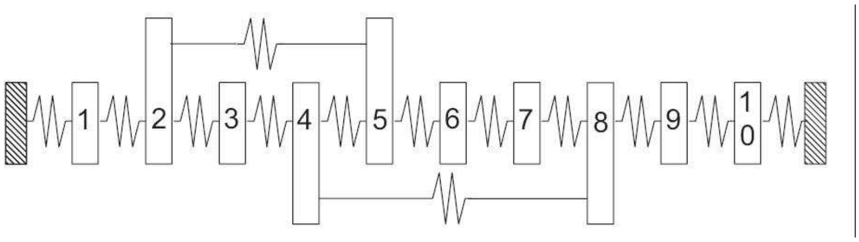

Figure 2: Mass-spring system.

$$K_0 = \begin{pmatrix} 2000 & -1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1000 & 3000 & -1000 & 0 & -1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1000 & 2000 & -1000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1000 & 3000 & -1000 & 0 & 0 & -1000 & 0 & 0 \\ 0 & -1000 & 0 & -1000 & 3000 & -1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1000 & 2000 & -1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1000 & 2000 & -1000 & 0 & 0 \\ 0 & 0 & 0 & -1000 & 0 & 0 & -1000 & 3000 & -1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1000 & 2000 & -1000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1000 & 2000 \end{pmatrix}.$$

The measured data for our experiment was simulated by reducing stiffness of the spring between masses 2 and 5 to 600 N/ m and adding Gaussian noise with s $\delta = 2\%$. The analytical eigenvalue and eigenvector matrices are:

$$\Lambda = \begin{pmatrix} -6.23 & 71.1 & 0 & 0 \\ -71.1 & -6.23 & 0 & 0 \\ 0 & 0 & -3.67 & 65.9 \\ 0 & 0 & -65.9 & -3.67 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0.142 & 0.001 & -0.161 & -0.001 \\ -0.438 & 0.020 & 0.372 & -0.050 \\ 0.288 & 0.065 & -0.056 & 0.031 \\ -0.502 & -0.206 & -0.191 & 0.087 \\ 0.479 & 0.148 & -0.296 & -0.034 \\ -0.136 & -0.011 & 0.263 & 0.091 \\ -0.066 & 0.011 & -0.346 & -0.145 \\ 0.339 & 0.003 & 0.599 & 0.063 \\ -0.122 & -0.007 & -0.296 & -0.093 \\ 0.040 & 0.010 & 0.115 & 0.075 \end{pmatrix}.$$

The measured eigenvalue and eigenvector matrices are:

$$\tilde{\Lambda} = \begin{pmatrix} -6.16 & 69.8 & 0 & 0 \\ -69.8 & -6.16 & 0 & 0 \\ 0 & 0 & -4.7 & 64.9 \\ 0 & 0 & -64.9 & -4.7 \end{pmatrix}, \quad \tilde{\Phi} = \begin{pmatrix} 0.102 & 0.026 & -0.172 & -0.023 \\ -0.283 & -0.061 & 0.401 & 0.023 \\ 0.282 & 0.115 & -0.195 & -0.005 \\ -0.579 & -0.240 & 0.074 & 0.068 \\ 0.341 & 0.242 & -0.354 & -0.202 \\ -0.067 & -0.054 & 0.286 & 0.242 \\ -0.168 & 0.036 & -0.249 & -0.340 \\ 0.508 & -0.042 & 0.362 & 0.382 \\ -0.207 & 0.009 & -0.183 & -0.246 \\ 0.077 & 0.011 & 0.060 & 0.130 \end{pmatrix}.$$

We first form matrices

$$\tilde{A} = \tilde{\Phi}\tilde{\Lambda}^2, \quad \tilde{B} = \tilde{\Phi}\tilde{\Lambda}, \quad \tilde{C} = \tilde{\Phi}, \quad \text{and} \quad E = -(M_0\tilde{A} + D_0\tilde{B} + K_0\tilde{C}).$$

Then applying Algorithm 1 proposed in Section 2, letting the initial iterative matrices

$$X_0 = zeros(4), \quad Y_0 = zeros(4), \quad Z_0 = zeros(4),$$

Table 4: The residual for Example 3.1.

$(\tilde{\lambda}_i, \tilde{\phi}_i)$	$(\tilde{\lambda}_1, \tilde{\phi}_1)$	$(\tilde{\lambda}_2, \tilde{\phi}_2)$	$(\tilde{\lambda}_3, \tilde{\phi}_3)$	$(\tilde{\lambda}_4, \tilde{\phi}_4)$
res$(\tilde{\lambda}_i, \tilde{\phi}_i)$	3.4864e-011	3.4864e-011	6.8775e-011	6.8775e-011

and iterating for 11 steps, we obtain the least norm solution (X^*, Y^*, Z^*) of the linear matrix equation

$$X\tilde{A} + Y\tilde{B} + Z\tilde{C} = E.$$

Finally the coefficients of the updated system can be expressed as

$$M_{\rm new} = X^* + M_0, \quad D_{\rm new} = Y^* + D_0, \quad K_{\rm new} = Z^* + K_0.$$

To save the space, we shall not report the data of these resulting updated matrices $M_{\rm new}$, $D_{\rm new}$, $K_{\rm new}$, but will make them available upon request. We merely show the bar graphs of the magnitude of the components of the matrices $D_0 - D_{\rm new}$, $K_0 - K_{\rm new}$. Similar graph exists for the matrix $M_0 - M_{\rm new}$.

We also define the residual as

$$res(\tilde{\lambda}_i, \tilde{\phi}_i) = \|(\tilde{\lambda}_i^2 M_{\rm new} + \tilde{\lambda}_i D_{\rm new} + K_{\rm new})\tilde{\phi}_i\|.$$

Table 4 shows the residual $res(\tilde{\lambda}_i, \tilde{\phi})$, where $(\tilde{\lambda}_i, \tilde{\phi})$ is the computed eigenpairs of $Q(\lambda)$, for $j = 1, 2, 3, 4$.

Therefore, the prescribed eigenvalues and eigenvectors are embedded in the model

$$(\tilde{\lambda}^2 M_{\rm new} + \tilde{\lambda} D_{\rm new} + K_{\rm new})v = 0,$$

and the updated matrices $M_{\rm new}$, $D_{\rm new}$, $K_{\rm new}$, which implies that the structural connectivity information of the analytical is preserved. Moreover, from Theorem 1, we know that $(M_{\rm new}, D_{\rm new}, K_{\rm new})$ is the optimal updated quadratic pencil.

Example 3.2. Updating of a statistically condensed oil rig model. Consider the model (M, D, K), where

- The matrices $M \in \mathbb{R}^{66\times 66}$ and $K \in \mathbb{R}^{66\times 66}$ come from the statistically condensed oil model of the *Harwell-Boeing* set **BCSSTRUC1** [24]. For simplicity, we let the analytical mass matrix $M_a = M$, the analytical stiffness matrix $K_a = K * 10^{-4}$, the two matrices are all symmetric positive-definite. The Frobenius norms of M_a and K_a are 66.0249 and 9.2845, respectively.
- The damping matrix D_a is defined by $D_a = \rho I_{66}$, with $\rho = 0.5$.

Because $M_a > 0$, the quadratic pencil $\lambda^2 M_a + \lambda D_a + K_a$ has 132 eigenpairs. Consider the given measured eigenvalues

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-0.4628,\ -0.5709,\ 0.3584,\ 0.2761\}.$$

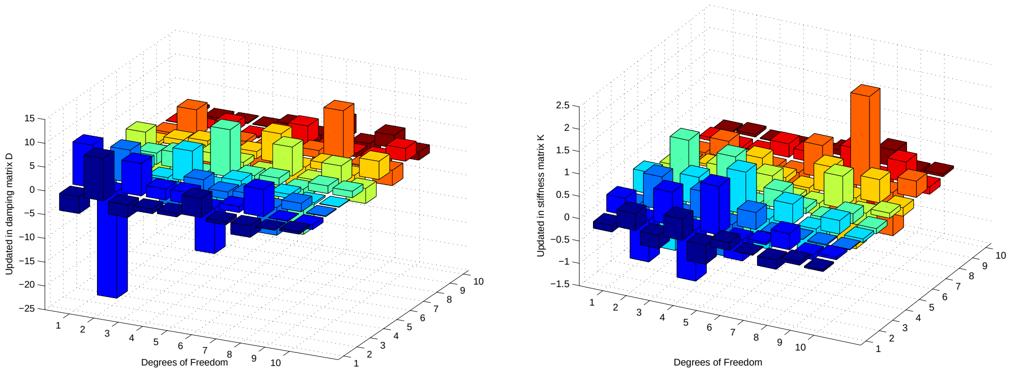

Figure 3: Magnitudes of the entries of the matrices $D_0 - D_{\text{new}}$, and $K_0 - K_{\text{new}}$.

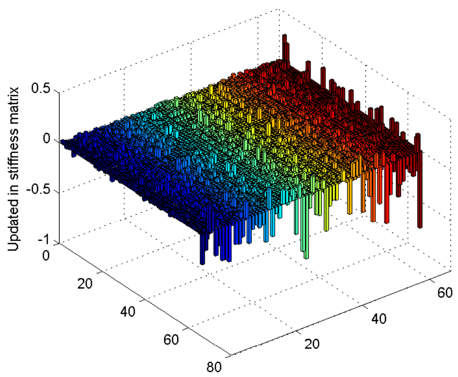

Figure 4: Magnitudes of the entries of the matrix $K_a - K_{\text{new}}$.

The eigenpairs of the experimental model are used to created the experimental modal date. It is assumed that only the fundamental mode characteristics are experimentally determined and only s (s<66) components of eigenvector are measured. Suppose now we are given the measured mode shapes $\Phi_j \in \mathbb{R}^{66}$, $j = 1, 2, 3, 4$. According to the proposed method in this paper, we can obtain the unique solution to the Model Updating Problem II, and it is easy to verify

$$\|M_a \Phi \Lambda^2 + D_{\text{new}} \Phi \Lambda + K_{\text{new}} \Phi\| = 2.5497 * 10^{-14}.$$

Therefore, the prescribed eigenvalues and eigenvectors have been embedded in the new model

$$(\lambda^2 M_a + \lambda D_{\text{new}} + K_{\text{new}})x = 0.$$

Fig. 4 shows the bar graphs of the magnitude of the components of the matrix $K - K_{\text{new}}$. Similar graphs exist for the matrix $D - D_{\text{new}}$.

4 Conclusions

Solving quadratic inverse eigenvalue problems for some partially prescribed eigeninformation is a challenging task in many applications. Many efforts have been made, theoretically and computationally. In the first part of this paper, we construct a new computationally efficient and symmetry preserving iterative method, based on the classical CG method, for the partially prescribed quadratic inverse eigenvalue problem. With the proposed algorithm the solvability of the problem can be determined automatically. If it is solvable, a desired solution can be obtained within finitely many steps. Some numerical examples show that the proposed algorithm is quite efficient.

Finite model updating has been a longstanding open problem for its many critical applications. To our knowledge, maintaining symmetry and reproduction of the measured data are the basic requirements for model updating. However, one of the most fundamental challenges is to require that the updating is made with minimal changes. Moody T. Chu called such updating the least squares model updating problem and pointed out that this is an area open for further research. In the second part of this paper, a new method, based on the algorithm proposed in the first part of this paper, for such model updating problem is proposed. The results of numerical experiments on updating a mass-spring system of 10 DoF and a statistically condensed oil rig model are presented to show the accuracy of the proposed method.

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