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A Maximum Entropy Method Based on Orthogonal Polynomials for Frobenius-Perron Operators

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Abstract. Let S: $[0,1]\to[0,1]$ be a chaotic map and let f^* be a stationary density of the Frobenius-Perron operator P_S: $L^1\to L^1$ associated with S. We develop a numerical algorithm for approximating f^*, using the maximum entropy approach to an under-determined moment problem and the Chebyshev polynomials for the stability consideration. Numerical experiments show considerable improvements to both the original maximum entropy method and the discrete maximum entropy method.

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Key words: Frobenius-Perron operator, stationary density, maximum entropy, orthogonal polynomials, Chebyshev polynomials.

1 Introduction

In the past fifty years, since the publication of the pioneering work of Jayne (see [8]), the idea of maximum entropy method has been widely applied to solving density function recovering problems in mathematical physics and stochastic analysis. This idea was first adopted in [4] to numerically compute a stationary density of a chaotic map S from the interval $[0,1]$ to itself, based on the classic Hausdorff moment problems.

The maximum entropy method developed in [4] has been applied to the computation of Lyapunov exponents of chaotic maps in [5], which is closely related to the computation of the stationary density f^* since the Lyapunov exponent can be calculated by

$$\lambda = \int_0^1 f^*(x) \ln |S'(x)| dx,$$

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that quantitatively describes the sensitivity of the orbits on the initial conditions for the chaotic dynamics. The numerical experiments in [4,5] suggest that for relatively small number of moments, the algorithm can produce better approximations of the stationary density and exact Lyapunov exponent than the famous Ulam's method (see [7,11]). But due to the ill-conditioning resulting from employing the standard monomial basis of $\{1, x, x^2, \cdots, x^n\}$ (the condition number may reach the order of 10^{17} for n=12), round-off errors dominate the computation of the algorithm even if a high precision Gauss quadrature is used in numerical integration.

Recently, the authors of [2] proposed a discrete version of a maximum entropy method for computing stationary densities and Lyapunov exponents. Basically they first approximate the Boltzmann entropy functional, which is the objective function of the maximum entropy optimization problem, by a high precision Gauss quadrature, and do the same thing for the moment constraints. The resulting optimization problem is still finite dimensional, but integration is avoided, which is natural since the Gauss quadrature numerical integration had been done before solving the discretized optimization problem. In their implementation of the algorithm, the monomial basis of polynomials is replaced with the Chebeshev polynomial basis. The computationally needed moments of the unknown stationary density with respect to the Chebeshev polynomials are estimated by the average values of the polynomials along the orbit of an initial point under the repeated iteration of the map S. This is justified in theory by the classic Birkhoff individual ergodic theorem, which says that the time average equals the space average for ergodic maps. As many as 150 moments can be used in [2] for the implementation of the algorithm. However, there is an approximation accuracy issue here, that is, some additional errors occur from approximating the Boltzmann entropy functional and the constraint equations. Such errors explain why a relatively large number of moments are needed for the numerical recovery of the stationary density to a prescribed precision.

In this paper, we intend to overcome the two main drawbacks of the original maximum entropy method for solving the stationary density problem of Frobenius-Perron operators. The first drawback is the ill-conditioning of the monomials, so we employ orthogonal polynomials in our numerical computation. The second drawback is related to the "homogeneous moment problem" proposed in [4] since the maximum entropy solution involves the underlying map which is only piecewise continuous in general. Thus, a good accuracy of the computed stationary density may not be guaranteed. To solve this problem, as is done in the paper [2], we use the same idea of Birkhoff's individual ergodic theorem, and consequently we solve a "nonhomogeneous moment problem" whose solution is a smooth function. Thus we propose a new practical algorithm for solving the stationary density problem of Frobenius-Perron operators, which combines the original idea of the maximum entropy method [4] and the idea of solving a nonhomogeneous moment problem [2], using the good stability property of the orthogonal polynomials. From the reported numerical experiment results one can see that the present algorithm can not only use as many moments as needed, but also give a faster convergence. For some maps our algorithm uses much