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On Higher Order Pyramidal Finite Elements

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Abstract. In this paper we first prove a theorem on the nonexistence of pyramidal polynomial basis functions. Then we present a new symmetric composite pyramidal finite element which yields a better convergence than the nonsymmetric one. It has fourteen degrees of freedom and its basis functions are incomplete piecewise triquadratic polynomials. The space of ansatz functions contains all quadratic functions on each of four subtetrahedra that form a given pyramidal element.

AMS subject classifications: 65N30

Key words: Pyramidal polynomial basis functions, finite element method, composite elements, three-dimensional mortar elements.

1 Introduction

Pyramidal mortar elements are very useful tools for making face-to-face connections between tetrahedral and block elements in the finite element method (see Fig. 1). This often occurs in joining tetrahedral meshes with hexahedral ones, which is common in many practical applications, where only part of the domain can be decomposed into block elements and the remainder of the domain, often near the boundary, is decomposed into tetrahedral elements. The first mortar elements were proposed by Zlámal. In [7], Zlámal introduced two kinds of triangular elements that enable us to connect the standard linear elements with the Hermite cubic elements.

Two types of composite incomplete trilinear and triquadratic pyramidal mortar elements with five (cf. Fig. 1) and thirteen degrees of freedom are presented in [6].

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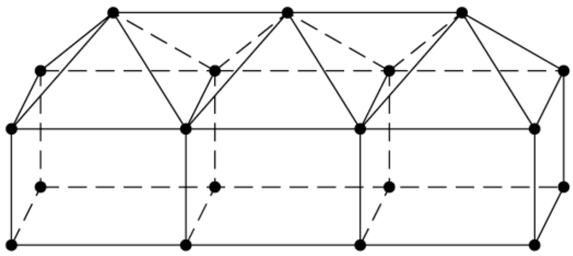

Figure 1: Mortar elements with 5 degree of freedoms.

Their basis functions are defined on the pyramidal element which is composed of two tetrahedra. This causes an artificial anisotropy in solving isotropic problems (compare with [4]). In [5] pyramidal elements composed of four tetrahedra are introduced which eliminate this artificial anisotropy. They also have five and thirteen degrees of freedom. Moreover, numerical results given in [5] indicate that an improvement of the convergence is obtained for both types of symmetric elements over the non-symmetric ones.

Wieners in [6] points out that it can be proven that it is not possible to define polynomial basis functions that are linear (or quadratic) on all the triangular faces and bilinear (or biquadratic) on the base of the pyramidal element. We will improve Wieners' result. We also develop a new composite piecewise triquadratic pyramidal element. To accomplish this we add a node at the center of the base of the element, thereby creating an element with fourteen degrees of freedom. This new pyramidal element, having nine nodes on the base, allows for a better connection to the common twenty-seven node triquadratic block element which has nine nodes on each face. Its space of ansatz functions contains all quadratic polynomials on each tetrahedron (from Figs. 2, 3 and 4).

This paper is organized as follows: in Section 2, we discuss the nonexistence of polynomial basis functions on a pyramidal element. We prove that there exists no continuously differentiable function on the pyramid (see Fig. 2), which would be linear on its four triangular faces and bilinear, but not linear, on its rectangular base. In Section 3.1 we present the basis functions for the two-tetrahedral composition of the pyramidal element (see Fig. 3) and prove that these functions meet the required criteria for basis functions. In Section 3.2 we take the average of these basis functions with their mirror images in order to derive a set of basis functions well defined on the four-tetrahedral composition of the pyramid (see Fig. 4). Finally, in Section 4 we present numerical results for both the two- and four-tetrahedral compositions.

2 Nonexistence of pyramidal polynomial basis functions

Let Co stand for the convex hull and let

$$\widehat{K} = \mathrm{Co}\big\{\widehat{A}_0, \widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \widehat{A}_4\big\} = \mathrm{Co}\big\{(0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1)\big\},$$

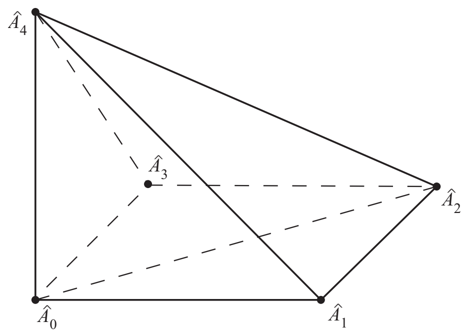

Figure 2: A reference pyramid.

be the reference pyramid (see Fig. 2). A general pyramid K can be obtained as the image of a nondegenerate linear affine mapping from $\widehat{K}$ into the three-dimensional space.

Wieners presents a theorem in [6] that states: "There exists no continuously differentiable conforming shape function for the pyramid which is linear, resp. bilinear on the faces." His theorem is not exactly formulated, since the function $p = 0$ satisfies all the assumptions of the theorem and p is continuously differentiable. Also his proof is not correct, as he considers only one special shape function p and not an arbitrary function satisfying all the assumptions of his theorem. Anyway, some of his ideas are quite sophisticated and we will employ them in the proof of the following theorem:

Theorem 2.1. *There exists no continuously differentiable function on the pyramid K that would be linear on its four triangular faces and bilinear, but not linear, on its rectangular base.*

Proof. First consider the reference pyramid $\widehat{K}$ and assume, to the contrary, that such a function $\widehat{f}$ exists. Since the vertices $\widehat{A}_i$, $i = 1,2,3,4$, are not contained in a plane, there exists exactly one linear polynomial $\widehat{p}$ such that

$$\widehat{p}(\widehat{A}_i) = \widehat{f}(\widehat{A}_i), \qquad i = 1,2,3,4.$$

Setting

$$\widehat{g} = \widehat{f} - \widehat{p},$$

on $\widehat{K}$, we see that $\widehat{g}$ satisfies all the assumptions of the theorem and vanishes on the faces $\widehat{A}_1\widehat{A}_2\widehat{A}_4$ and $\widehat{A}_2\widehat{A}_3\widehat{A}_4$. Therefore, the following derivatives in three linearly independent directions along edges vanish:

$$(0,-1,1)^\top \mathrm{grad}\,\widehat{g}(\widehat{A}_4) = (-1,0,1)^\top \mathrm{grad}\,\widehat{g}(\widehat{A}_4) = (-1,-1,1)^\top \mathrm{grad}\,\widehat{g}(\widehat{A}_4) = 0,$$

which implies that

$$\mathrm{grad}\,\widehat{g}(\widehat{A}_4) = 0. \tag{2.1}$$

By the assumptions, $\widehat{f}$ is not linear on the base and thus, the same has to be true also for $\widehat{g}$. Consequently, the equalities

$$\widehat{g}(\widehat{A}_i) = 0, \quad \text{for } i = 1,2,3,4,$$

imply that

$$\widehat{g}(\widehat{A}_0) \neq 0.$$

From this we find that

$$(0,0,-1)^\top \operatorname{grad} \widehat{g}(\widehat{A}_4) = \widehat{g}(\widehat{A}_4) - \widehat{g}(\widehat{A}_0) \neq 0,$$

because $\widehat{g}$ is linear on the edge $\widehat{A}_0\widehat{A}_4$. However, this contradicts Eq. (2.1).

For a general pyramid K it is enough to apply a linear affine transformation F_K from the reference pyramid $\widehat{K}$ to K to obtain a similar contradiction. $\square$

As linear and bilinear functions are quadratic, we also cannot construct polynomial elements on the whole pyramidal element whose ansatz functions are quadratic on triangular faces and biquadratic, but not quadratic, on the rectangular base, i.e., again we have to use composite elements.

3 Piecewise triquadratic pyramidal finite element

As in [5] the proposed piecewise triquadratic basis functions are defined on another reference pyramid which is different from the one in Section 2 (see Fig. 3):

$$\tilde{K} = \text{Co}\{A_0, A_1, A_2, A_3, A_4\} = \text{Co}\big\{(-1,-1,0),(1,-1,0),(1,1,0),(-1,1,0),(0,0,1)\big\},$$

where Co denotes the convex hull.

We will first present basis functions for the two-tetrahedral element composition. Then, following a process similar to that in [5] we take the average of these basis functions with their mirror images across the $x = 0$ plane in order to develop basis functions defined for the four-tetrahedral composition of the pyramidal element.

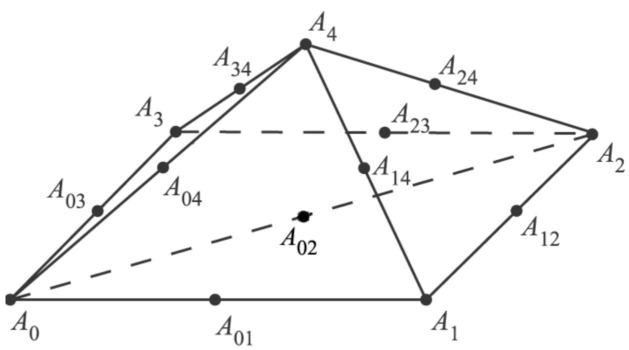

Figure 3: The piecewise triquadratic basis functions are defined on a reference pyramid.

3.1 Nonsymmetric composite pyramidal element

Employing a procedure analogous to that outlined in [5] we introduce the following basis functions defined on the reference pyramid $\tilde{K}$:

$$q_0(x,y,z)=\begin{cases}\frac{1}{4}(x+z)(x+z-1)(y-z-1)(y-z), & \text{for } x>y,\\ \frac{1}{4}(y+z)(y+z-1)(x-z-1)(x-z), & \text{for } x\le y,\end{cases}$$

$$q_1(x,y,z)=\begin{cases}-\frac{1}{4}(x+z)(y-z)[(x+z+1)(-y+z+1)-4z]-z(x-y), & \text{for } x>y,\\ -\frac{1}{4}(x-z)(y+z)(x-z+1)(-y-z+1), & \text{for } x\le y,\end{cases}$$

$$q_2(x,y,z)=\begin{cases}\frac{1}{4}(y-z)(x+z)(y-z+1)(x+z+1), & \text{for } x>y,\\ \frac{1}{4}(y+z)(x-z)(x-z+1)(y+z+1), & \text{for } x\le y,\end{cases}$$

$$q_3(x,y,z)=\begin{cases}\frac{1}{4}(x+z)(y-z)(y-z+1)(x+z-1), & \text{for } x>y,\\ \frac{1}{4}(x-z)(y+z)[(x-z-1)(y+z+1)+4z]+z(x-y), & \text{for } x\le y,\end{cases}$$

$$q_4(x,y,z)=z(2z-1),$$

$$q_{01}(x,y,z)=\begin{cases}-\frac{1}{2}(x+z-1)[((y-z-1)(x+1)y-z)+z(2x+1)], & \text{for } x>y,\\ -\frac{1}{2}(x-z+1)(y+z-1)(x-1)y, & \text{for } x\le y,\end{cases}$$

$$q_{12}(x,y,z)=\begin{cases}-\frac{1}{2}(y-z+1)[((x+z+1)(y-1)x-z)+z(2y+1)], & \text{for } x>y,\\ -\frac{1}{2}(x-z+1)(y+z-1)(y+1)x, & \text{for } x\le y,\end{cases}$$

$$q_{23}(x,y,z)=\begin{cases}-\frac{1}{2}(y-z+1)(x+z-1)(x+1)y, & \text{for } x>y,\\ -\frac{1}{2}(x-z+1)[((y+z+1)(x-1)y-z)+z(2x+1)], & \text{for } x\le y,\end{cases}$$

$$q_{03}(x,y,z)=\begin{cases}-\frac{1}{2}(y-z+1)(x+z-1)(y-1)x, & \text{for } x>y,\\ -\frac{1}{2}(y+z-1)[((x-z-1)(y+1)x-z)+z(2y+1)], & \text{for } x\le y,\end{cases}$$

$$q_{02}(x,y,z)=\begin{cases}(y-z+1)(x+z-1)[(y-1)(x+1)+z(x-y+z+1)], & \text{for } x>y,\\ (x-z+1)(y+z-1)[(y+1)(x-1)-z(x-y-z-1)], & \text{for } x\le y,\end{cases}$$

$$q_{04}(x,y,z)=\begin{cases}z(x+z-1)(y-z-1), & \text{for } x>y,\\ z(y+z-1)(x-z-1), & \text{for } x\le y,\end{cases}$$

$$q_{14}(x,y,z)=\begin{cases}-z[(x+z+1)(y-z-1)+4z], & \text{for } x>y,\\ -z(x-z+1)(y+z-1), & \text{for } x\le y,\end{cases}$$

$$q_{24}(x,y,z)=\begin{cases}z(y-z+1)(x+z+1), & \text{for } x>y,\\ z(x-z+1)(y+z+1), & \text{for } x\le y,\end{cases}$$

$$q_{34}(x,y,z)=\begin{cases}-z(y-z+1)(x+z-1), & \text{for } x>y,\\ -z[(y+z+1)(x-z-1)+4z], & \text{for } x\le y,\end{cases}$$

where q_{km} corresponds to node A_{km} located at the midpoint of the edge A_kA_m.

Theorem 3.1. *The basis functions $q_0,\cdots,q_4,q_{01},\cdots,q_{34}$ satisfy the following conditions:*

1) $q_i(A_j)=\delta_{ij}$, $i,j\in\{0,1,2,3,4,01,12,23,03,02,04,14,24,34\}$.

2) Each basis function is biquadratic on the base of $\tilde{K}$ (given by the equation $z=0$).

3) Each basis function is quadratic on all triangular faces of $\tilde{K}$.

4) Each basis function is continuous in the interelement boundary of $\tilde{K}$ (given by the equation $x = y$).

5) Each quadratic polynomial can be expressed as a linear combination of the fourteen basis functions $q_0, \cdots, q_4, q_{01}, \cdots, q_{34}$ on the subtetrahedron $\tilde{K}_1 = Co\{A_0, A_1, A_2, A_4\}$, as well as on $\tilde{K}_2 = Co\{A_0, A_2, A_3, A_4\}$.

Proof. 1) This can be easily verified by a direct calculation.

2) Setting $z = 0$, we immediately find that $q_0, \cdots, q_4, q_{01}, \cdots, q_{34}$ are biquadratic, for instance

$$q_0(x, y, 0) = \frac{x(x-1)y(y-1)}{4}, \qquad \text{for } x > y.$$

3) (i) On the face $A_0A_1A_4$, which is contained in the plane $z = 1 + y$, we have

$$\begin{aligned}
&q_2(x > y) = q_3(x > y) = q_{12}(x > y) = q_{23}(x > y) = 0,\\
&q_{03}(x > y) = q_{02}(x > y) = q_{24}(x > y) = q_{34}(x > y) = 0,\\
&q_0(x > y) = \frac{1}{2}(x + y + 1)(x + y), \quad q_1(x > y) = \frac{1}{2}(x - y - 1)(x - y),\\
&q_4(x > y) = (y + 1)(2y + 1), \qquad\quad q_{01}(x > y) = (x + y)(-x + y),\\
&q_{04}(x > y) = -2(y + 1)(x + y), \qquad q_{14}(x > y) = 2(y + 1)(x - y).
\end{aligned}$$

(ii) On the face $A_1A_2A_4$, which is contained in the plane $z = 1 - x$, we have

$$\begin{aligned}
&q_0(x > y) = q_3(x > y) = q_{01}(x > y) = q_{23}(x > y) = 0,\\
&q_{03}(x > y) = q_{02}(x > y) = q_{04}(x > y) = q_{34}(x > y) = 0,\\
&q_1(x > y) = \frac{1}{2}(x - y - 1)(x - y), \quad q_2(x > y) = \frac{1}{2}(x + y - 1)(x + y),\\
&q_4(x > y) = (x - 1)(2x - 1), \qquad\quad q_{12}(x > y) = (x + y)(x - y),\\
&q_{14}(x > y) = 2(1 - x)(x - y), \qquad q_{24}(x > y) = 2(1 - x)(x + y).
\end{aligned}$$

(iii) On the face $A_2A_3A_4$, which is contained in the plane $z = 1 - y$, we have

$$\begin{aligned}
&q_0(x \le y) = q_1(x \le y) = q_{01}(x \le y) = q_{12}(x \le y) = 0,\\
&q_{03}(x \le y) = q_{02}(x \le y) = q_{04}(x \le y) = q_{14}(x \le y) = 0,\\
&q_2(x \le y) = \frac{1}{2}(x + y - 1)(x + y), \quad q_3(x \le y) = \frac{1}{2}(x - y + 1)(x - y),\\
&q_4(x \le y) = (1 - y)(1 - 2y), \qquad\quad q_{23}(x \le y) = (x + y)(-x + y),\\
&q_{24}(x \le y) = 2(1 - y)(x + y), \qquad q_{34}(x \le y) = 2(1 - y)(-x + y).
\end{aligned}$$

(iv) On the face $A_0A_3A_4$, which is contained in the plane $z = 1 + x$, we have

$$\begin{aligned}
&q_1(x \le y) = q_2(x \le y) = q_{01}(x \le y) = q_{12}(x \le y) = 0,\\
&q_{23}(x \le y) = q_{02}(x \le y) = q_{14}(x \le y) = q_{24}(x \le y) = 0,\\
&q_0(x \le y) = \frac{1}{2}(x + y + 1)(x + y), \quad q_3(x \le y) = \frac{1}{2}(x - y + 1)(x - y),
\end{aligned}$$

$$q_4(x \le y) = (x+1)(2x+1), \qquad q_{03}(x \le y) = (x+y)(x-y),$$
$$q_{04}(x \le y) = 2(-1-x)(x+y), \qquad q_{34}(x \le y) = 2(1+x)(-x+y).$$

From (i)-(iv) we observe that each q_i is quadratic on any triangular face of $\tilde{K}$.

4) By setting $x = y$, we can easily see that the functions $q_0, \cdots, q_4, q_{01}, \cdots, q_{34}$ are continuous in the plane $x = y$.

5) The proof is quite simple, but long and technical. Consider, for instance, the quadratic function

$$p(x,y,z) = \frac{(y-z+1)(y-z)}{2}.$$

We show that it can be expressed as a linear combination of basis functions on $\tilde{K}_1$. Let

$$p = \sum_i c_i q_i, \tag{3.1}$$

on $\tilde{K}_1$. Then by the property 1) of Theorem 3.1, we have

$$p(A_j) = \sum_i c_i q_i(A_j) = c_j q_j(A_j) = c_j. \tag{3.2}$$

Since

$$p(A_0) = p(A_1) = p(A_4) = p(A_{01}) = p(A_{04}) = p(A_{14}) = p(A_{02}) = p(A_{12}) = p(A_{24}) = 0,$$

we get

$$c_0 = c_1 = c_4 = c_{01} = c_{04} = c_{14} = c_{02} = c_{12} = c_{24} = 0.$$

Moreover, since $p(A_2) = 1$, we get from Eq. (3.2) that $c_2 = 1$. Thus, Eq. (3.1) reduces to

$$p = q_2 + c_3 q_3 + c_{03} q_{03} + c_{23} q_{23} + c_{34} q_{34},$$

on $\tilde{K}_1$. From this and the definition of q_i for $x \ge y$ we get

$$\begin{aligned}
&2(y-z+1)(y-z) - (y-z)(x+z)(y-z+1)(x+z+1)\\
=&c_3(x+z)(y-z)(y-z+1)(x+z-1) - 2c_{03}(y-z+1)(x+z-1)(y-1)x\\
&- 2c_{23}(y-z+1)(x+z-1)(x+1)y - 4c_{34}z(y-z+1)(x+z-1)\\
=&(y-z+1)(x+z-1)\big[c_3(x+z)(y-z) - 2c_{03}x(y-1) - 2c_{23}(x+1)y - 4c_{34}z\big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&(y-z)\big[2-(x+z)(x+z+1)\big]\\
=&(x+z-1)\big[c_3(x+z)(y-z) - 2c_{03}x(y-1) - 2c_{23}(x+1)y - 4c_{34}z\big].
\end{aligned}$$

For $x = z = 0$, we get $c_{23} = 1$, for $y = z = 0$, we get $c_{03} = 0$, and for $x = y = 0$, we find that

$$-z\big[2 - z(z+1)\big] = (z-1)\big[-c_3 z^2 - 4c_{34}z\big],$$

i.e., $c_3 = -1$ and $c_{34} = 1/2$.

A similar calculation can be made for other quadratic functions from a given basis and also on $\tilde{K}_2$. $\square$

3.2 Symmetric composite pyramidal element

In order to eliminate the artificial anisotropy occurring in the two-tetrahedral composition we now take the average of the basis functions

$$q_i, i \in \{0,1,2,3,4,01,12,23,03,02,04,14,24,34\},$$

with their mirror images. Namely, we consider a mirror image mapping

$$M : (x,y,z) \mapsto (-x,y,z),$$

and define

$$\begin{aligned}
&\overline{q}_0(x,y,z) = q_1(-x,y,z), && \overline{q}_1(x,y,z) = q_0(-x,y,z), && \text{(3.3a)}\\
&\overline{q}_2(x,y,z) = q_3(-x,y,z), && \overline{q}_3(x,y,z) = q_2(-x,y,z), && \text{(3.3b)}\\
&\overline{q}_{01}(x,y,z) = q_{01}(-x,y,z), && \overline{q}_{12}(x,y,z) = q_{03}(-x,y,z), && \text{(3.3c)}\\
&\overline{q}_{23}(x,y,z) = q_{23}(-x,y,z), && \overline{q}_{03}(x,y,z) = q_{12}(-x,y,z), && \text{(3.3d)}\\
&\overline{q}_{02}(x,y,z) = q_{02}(-x,y,z), && \overline{q}_{04}(x,y,z) = q_{14}(-x,y,z), && \text{(3.3e)}\\
&\overline{q}_{14}(x,y,z) = q_{04}(-x,y,z), && \overline{q}_{24}(x,y,z) = q_{34}(-x,y,z), && \text{(3.3f)}\\
&\overline{q}_{34}(x,y,z) = q_{24}(-x,y,z). && && \text{(3.3g)}
\end{aligned}$$

Setting

$$\begin{aligned}
&m_i = \frac{q_i + \overline{q}_i}{2}, \quad \text{for } i \in \{0,1,2,3,01,12,23,03,02,04,14,24,34\}, && \text{(3.4a)}\\
&m_4 = q_4. && \text{(3.4b)}
\end{aligned}$$

Theorem 3.2. *The basis functions $m_0, \cdots, m_4, m_{01}, \cdots, m_{34}$ satisfy the following conditions:*

1) $m_i(A_j) = \delta_{ij}$, $i,j \in \{0,1,2,3,4,01,12,23,03,02,04,14,24,34\}$.

2) Each basis function is biquadratic on the base of $\tilde{K}$ (given by the equation $z = 0$).

3) Each basis function is quadratic on all triangular faces of $\tilde{K}$.

4) Each basis function is continuous in the interelement boundaries of $\tilde{K}$ (given by the equations $x = y$ and $x = -y$).

5) Each quadratic polynomial can be expressed as a linear combination of the fourteen basis functions $m_0, \cdots, m_4, m_{01}, \cdots, m_{34}$ on each of the four subtetrahedra whose union is the original pyramid.

The proof of 1)-4) is an immediate consequence of Eqs. (3.3)-(3.4), and Theorem 3.1. The proof of 5) can be done similarly to Theorem 3.1.

The mirror image mapping has the effect of creating a pyramidal element composed of four tetrahedra (the pyramid is split along the $x = y$ and $x = -y$ planes, see Fig. 4). In addition, this averaging helps us in reducing the discretization error coming from anisotropy (see [5]), since the new basis functions are now more symmetric on $\tilde{K}$.

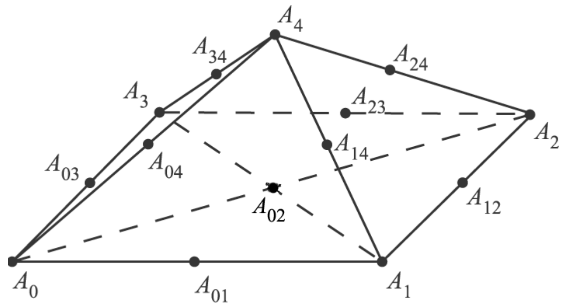

Figure 4: A pyramidal element.

4 Numerical experiments

In order to test the proposed basis functions, experiments were conducted to solve Poisson's equation with Dirichlet boundary conditions

$$-\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\Big)=f, \quad \text{in } \Omega, \tag{4.1a}$$

$$u=0, \quad \text{on } \partial\Omega, \tag{4.1b}$$

where $\Omega=(0,1)\times(0,1)\times(0,1)$.

The true solution used for the problem is oscillating:

$$u(x,y,z)=\sin(\pi x)\sin(2\pi y)\sin(3\pi z).$$

The overall mesh consisted of $N\times N\times N$ small cubes, where each small cube is constructed from six pyramidal elements containing a common vertex in the centre of each cube (see Fig. 5). Experiments were carried out using the basis functions both before and after the mirror image mappings were made. In the former case the pyramidal element was split into two tetrahedra along the $x=y$ plane. In the second case the element was split into four tetrahedra along the $x=y$ and $x=-y$ planes and the mirror image mappings were applied. Denote by u_h and $\tilde{u}_h$ the finite element solution of (4.1) before and after symmetrization, respectively. In both cases Gaussian


Figure 5: The overall mesh with small cubes.

Table 1: Discretization errors before and after symmetrization.

$h=1/N$	$6N^3$	$\|u-u_h\|_{L_2}$	$\frac{\|u-u_{2h}\|_{L_2}}{\|u-u_h\|_{L_2}}$	$\|u-\tilde{u}_h\|_{L_2}$	$\frac{\|u-\tilde{u}_{2h}\|_{L_2}}{\|u-\tilde{u}_h\|_{L_2}}$
1/4	384	1.61e-3	NONE	1.58e-3	NONE
1/8	3072	3.77e-4	4.27	3.69e-4	4.28
1/16	24576	9.23e-5	4.08	9.04e-5	4.08
1/32	196608	2.30e-5	4.01	2.25e-5	4.02
1/64	1572864	5.73e-6	4.01	5.61e-6	4.01

cubature with 11 integration points (cf. [1–3]) were used for the numerical integration over the tetrahedral components. For each $N \in \{4, 8, 16, 32, 64\}$, we present in Table 1 the number of pyramidal elements, the L_2-norms of the discretization errors and the ratios

$$r = \frac{\|u-u_{2h}\|_{L_2}}{\|u-u_h\|_{L_2}}, \quad \text{and} \quad \tilde{r} = \frac{\|u-\tilde{u}_{2h}\|_{L_2}}{\|u-\tilde{u}_h\|_{L_2}}.$$

We observe a slight improvement in accuracy of the solution $\tilde{u}_h$, since the symmetrized elements reduce the error coming from the anisotropy of the mesh. In the future, we plan to use suitable higher order integration formulae to increase the practical convergence rate.

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