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Spectral Element Discretization of the Stokes Equations in Deformed Axisymmetric Geometries

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Abstract. In this paper, we study the numerical solution of the Stokes system in deformed axisymmetric geometries. In the azimuthal direction the discretization is carried out by using truncated Fourier series, thus reducing the dimension of the problem. The resulting two-dimensional problems are discretized using the spectral element method which is based on the variational formulation in primitive variables. The meridian domain is subdivided into elements, in each of which the solution is approximated by truncated polynomial series. The results of numerical experiments for several geometries are presented.

AMS subject classifications: 65N35, 65N55

Key words: Spectral element method, Stokes equations, variational formulation, deformed geometries, Fourier expansion.

1 Introduction

In many problems of fluid mechanics one encounters flows in domains with axisymmetric geometries. Typical examples of such flows are the blood flow in vessels, hydrological problems (flow in pipelines), etc (see, e.g., [1] for more examples). As is well-documented [9], in axisymmetric geometries the solutions of the governing partial differential equations (PDEs) admit a Fourier expansion with respect to the angular (azimuthal) variable. The Fourier coefficients in this expansion are solutions of an infinite system of two-dimensional problems in the meridian domain. Such an expansion allows for a reduction of the dimension of the problem. In [2, 3] this idea

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was developed to derive an efficient strategy for solving the Stokes and Navier-Stokes equations with a finite element based discretization.

The extension of these ideas to spectral methods is non-trivial because of the complex nature of the meridian domains which, in this case, is a greater constraint in comparison to the finite element approach. In general, even after the decomposition of the meridian domain into simpler spectral elements, one has to deal with trapezoidal and curved sub-domains. In [15], it is shown that the spectral accuracy of spectral methods in such domains is preserved by applying the idea of *over-integration*. The aim of this paper is to combine this idea with a spectral element discretization to problems in *deformed* axisymmetric geometries, i.e., axisymmetric domains containing trapezoidal or curved parts. In particular, we study and develop a numerical procedure for solving the Stokes equations in deformed axisymmetric domains. This is achieved by the reduction of the dimension of the problem, where the full three-dimensional problem is replaced by an infinite system of two-dimensional problems, which, in turn, is approximated by a finite system of such problems. Each two-dimensional problem is solved with the spectral element method. We consider several geometries the discretization of which by spectral methods is non-trivial.

In particular, let Ω be a bounded connected domain in $\mathbb{R}^3$, which is invariant under rotation around the z-axis. Also, assume that $\partial\Omega$ is the boundary of Ω. We consider the Stokes problem for an incompressible fluid

$$\begin{cases} -\Delta \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}$ is the velocity and p is the pressure of the fluid. In Problem (1.1), the external force $\boldsymbol{f}$ and $\boldsymbol{g}$ are given functions. We will further assume the usual flux condition

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n}\, d\tau = 0, \tag{1.2}$$

where $\boldsymbol{n}$ denotes the outward normal unit vector to $\partial\Omega$.

The paper is organized as follows. In Section 2 we introduce the Fourier expansion of Problem (1.1) and the functional (weighted) spaces necessary for the correct mathematical setting of the problem. We subsequently, provide the variational formulation for two-dimensional problems corresponding to the various Fourier modes. Section 3 is devoted to the spectral element discretization. In Section 4, we present the numerical procedure for the solution of the discrete problem. Finally, in Section 5 we present the results of several numerical experiments.

2 Variational formulation

2.1 Fourier expansion

Let (x, y, z) denote a set of Cartesian coordinates in $\mathbb{R}^3$ such that Ω is invariant under rotation around the z-axis. The polar coordinates (r, ϑ, z), where $r \geq 0$ and $-\pi \leq \vartheta \leq$