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Nonlinear Axisymmetric Deformation Model for Structures of Revolution

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Abstract. An axisymmetric formulation for modeling three-dimensional deformation of structures of revolution is presented. The axisymmetric deformation model is described using the cylindrical coordinate system. Large displacement effects and material nonlinearities and anisotropy are accommodated by the formulation. Mathematical derivation of the formulation is given, and an example is presented to demonstrate the capabilities and efficiency of the technique compared to the full three-dimensional model.

AMS subject classifications: 74B20, 74G15

Key words: Modeling, axisymmetry, nonlinear elasiticity, large deformations, numerical simulations.

1 Introduction

A mathematical model is a way of describing a realistic phenomenon by mathematical tools. Many of these phenomena originate from physical problems. Most physical problems are naturally formulated as initial boundary value problems in three-dimensional domains (3D). With today's techniques, 3D computations are still very expensive due to the number of unknowns and the complexity of the governing partial differential equations. The geometry of the domain is also an important source of algorithmic complexity, due to the representation of the surface and the mesh design inside the domain. Reducing the analysis of a 3D structure to two-dimensional domains (2D) provides a great convenience and efficiency compared to the full 3D analysis. In some cases, a simplified 2D model can be obtained from the 3D model by, for instance, neglecting or integrating with respect to one of the domain dimensions. This leads to only a 2D approximation model for the full 3D model. On the other hand, an intermediate situation between the full 3D geometry and the plane 2D reduction

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can be obtained. In order to perform such a reduction, however, the structure should have an axisymmetric shape. Moreover, additional assumptions of axisymmetry on the material parameters, the data and the solution are required (this is often the case for physical or mechanical systems of equations). If such assumptions of axisymmetry hold, then the resulting 2D model is called an axisymmetric model. Besides its instructional value, the treatment of axisymmetric structures has considerable practical interest in aerospace, civil, mechanical and nuclear engineering.

In the literature, a large number of axisymmetric models have been proposed. Among many other studies, for instance, Jeon et al. [12] proposed an axisymmetric model of the dome tendons in nuclear containment building. An axisymmetric finite element model has been developed by Suzuki and Maruyama [19] to evaluate earthquake responses of seismically isolated tunnels. Gallouet and Herbin [7] developed an axisymmetric cell centered finite volume scheme in order to numerically simulate the diffusion and assimilation by photosynthesis of CO_2 within a leaf. Deparis [3] studied numerical methods for an axisymmetric problem of fluid-structure interface with application to blood flow. In the framework of continuum mechanics, Wang Min and Tian You [20] developed an elastic axisymmetric model for quasi-crystal cubes. Eftaxiopoulos and Atkinson [4] treated an axisymmetric anisotropic elastic model for the angioplastic balloon. Bernardi et al. [2] developed the axisymmetric deformation model in the case of linear elasticity and small displacements.

In the present paper, we study the deformation of a structure of revolution in the framework of nonlinear elasticity and large deformations. The considered 3D computation domain is supposed invariant by rotation around an axis. Such a domain is generated by rotating a 2D set, the meridian domain, around the axis. Under some assumptions of axisymmetry on the material constitutive law and the loading, we derive the axisymmetric model as both a nonlinear system of partial differential equations and a variational problem written on the 2D meridian domain. The constitutive law can be nonlinear and anisotropic involving variable fiber direction through the material. Moreover, the main advantage of the proposed model is the fact that the deformation of the 2D meridian domain is done in the 3D space, hence allowing twisting during deformation. Therefore, the axisymmetric model considered in this paper provides a bridge to the treatment of three-dimensional nonlinear anisotropic elasticity. It is worth mentioning that in the linear framework, a structure of revolution under non-axisymmetric loading can be treated by a Fourier decomposition method. This involves decomposing the load into a Fourier series in the circumferential direction, calculating the response of the structure to each harmonic term retained in the series, and superposing the results, see Bernardi et al. [2]. The axisymmetric problem considered in this paper may be viewed as computing the response to the zero-th harmonic. This superposition technique, however, is limited to linear problems.

The main goal of the present paper is to derive and numerically validate the nonlinear axisymmetric deformation model. However, the problem of existence and uniqueness of solutions of the proposed axisymmetric model is very hard and is not addressed in the present paper. In some particular simplified similar problems, closed form solutions have been obtained. For instance, in the framework of isotropic linear elasticity, the solution of the problem of the large deformation of axial symmetry circular membrane under central force has been obtained, see Shanlin and Zhoulian [17] and, Hao and Yan [10]. On the other hand, Gao and Ogden [8] have proved the existence of solutions of the pure azimuthal shear problem for a circular cylindrical tube of nonlinearly elastic material, both isotropic and anisotropic. Moreover, for particular choices of the strain energy function, one convex and one nonconvex, closed-form solutions have been obtained.

By comparison to the existing axisymmetric models, the principal advantages of the proposed axisymmetric deformation model presented in this paper are: (1) the proposed model deals with anisotropic nonlinear constitutive laws regardless the strain energy function is convex or nonconvex and, (2) it reduces the problem of large deformation of 3D volume bodies of revolution to a mathematical problem of partial differential equations defined over a 2D domain. For instance, in all of the problems studied in [8, 10] and [17], the considered domains are not 3D volume domains and their corresponding reduced problems are one-dimensional.

The paper is presented as follows: first the full 3D deformation model is given, then the geometry is described and some of the tools are recalled. In the fourth section, the required axisymmetric assumptions on the loading, the data and the solution are presented, then the axisymmetric mechanical model is derived. Linearization and solving technique are proposed in Section 5. In the last section, numerical simulations of the deformation of an axisymmetric model of the left ventricle are shown. Moreover a comparison, in terms of complexity in time and computational cost, of the axisymmetric and the full 3D deformation models is presented. Also the discrete L^∞ and L^2 norms of the absolute and the relative errors of displacements that correspond to five different values of the applied external pressure are shown.

2 Three-dimensional framework

Let $\mathbb{R}^3$ be three-dimensional space and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an orthonormal basis where $\boldsymbol{e}_3$ represents the vertical direction.

2.1 Reference and actual configurations

Let Ω_R be an open, bounded, connected, piecewise C^1-regular set of $\mathbb{R}^3$. The physical reference configuration that will be subjected to deformation corresponds to the set $\overline{\Omega}_R$. We denote by Γ_R the boundary of the domain Ω_R. We suppose that Γ_R is the union of two parts:

$$\Gamma_R = \overline{\Gamma}_{R0} \cup \overline{\Gamma}_{R1}, \quad \text{where } \Gamma_{R0} \cap \Gamma_{R1} = \phi.$$

Indeed, in the sequel of the paper, the boundary Γ_{R0} will be subjected to a Dirichlet boundary condition and the boundary Γ_{R1} will be subjected to a Neumann (pressure) boundary condition. A material point in the reference configuration will be identified by its Cartesian coordinates $X = (X_1, X_2, X_3)$ also called the Lagrangian variable.

A deformation is a one-to-one mapping φ from the reference configuration $\overline{\Omega}_R$ into $\mathbb{R}^3$,

$$\varphi : \overline{\Omega}_R \mapsto \mathbb{R}^3, \quad \text{such that } \det(\nabla_X \varphi) > 0,$$

where ∇_X represents the gradient of φ with respect to the Lagrangian variable X. The actual (deformed) configuration is denoted by $\overline{\Omega}_\varphi = \varphi(\overline{\Omega}_R)$ where we suppose that the boundary Γ_φ of Ω_φ satisfies $\Gamma_\varphi = \varphi(\Gamma_R)$. Let

$$\Gamma_{\varphi 0} = \varphi(\Gamma_{R0}) \quad \text{and} \quad \Gamma_{\varphi 1} = \varphi(\Gamma_{R1}).$$

We have

$$\Gamma_\varphi = \overline{\Gamma}_{\varphi 0} \cup \overline{\Gamma}_{\varphi 1}.$$

A material point in the actual configuration will be identified by its Cartesian coordinates $x = (x_1, x_2, x_3)$ also called the Eulerian variable.

2.2 Balance equations

In continuum mechanics, a solid body in its actual configuration at equilibrium is subjected to two types of applied forces:

- applied volume forces (forces distributed over the volume), $\boldsymbol{f}_\varphi$.
- applied surface forces (forces distributed over the surface), $\boldsymbol{g}_\varphi$.

The equilibrium system in Cartesian coordinates of the actual configuration Ω_φ is given by:

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}_\varphi, \qquad \text{in } \Omega_\varphi, \tag{2.1a}$$
$$\boldsymbol{\sigma} \boldsymbol{n}_\varphi = \boldsymbol{g}_\varphi, \qquad \text{on } \Gamma_{\varphi 1}, \tag{2.1b}$$

where $\boldsymbol{\sigma}$ represents the Cauchy stress tensor and $\boldsymbol{n}_\varphi$ is the outward unit normal to $\Gamma_{\varphi 1}$. The tensor $\boldsymbol{\sigma}$ is related to φ through the constitutive law which will be described in the next section.

The boundary $\Gamma_{\varphi 0} = \varphi(\Gamma_{R0})$ is not subjected to any applied force. In fact, we impose a Dirichlet boundary condition on the boundary Γ_{R0}, i.e., we suppose that there exists a mapping φ_0 such that

$$\Gamma_{\varphi 0} = \varphi(\Gamma_{R0}) = \varphi_0(\Gamma_{R0}).$$

In other words, we write

$$\varphi = \varphi_0, \quad \text{on } \Gamma_{R0}.$$

The unknown of our problem is the mapping φ. The weak formulation associated to System (2.1) is given by: find

$$\varphi : \Omega_R \mapsto \Omega_\varphi, \qquad \varphi = \varphi_0, \quad \text{on } \Gamma_{R0},$$

and

$$\forall \boldsymbol{v}_\varphi : \overline{\Omega}_\varphi \mapsto \mathbb{R}^3, \quad \boldsymbol{v}_\varphi = 0, \quad \text{on } \Gamma_{\varphi 0},$$

such that

$$-\int_{\Omega_\varphi} \boldsymbol{\sigma} : \nabla_x \boldsymbol{v}_\varphi \, dx + \int_{\Omega_\varphi} \boldsymbol{f}_\varphi \cdot \boldsymbol{v}_\varphi \, dx + \int_{\Gamma_{\varphi 1}} \boldsymbol{g}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = 0, \tag{2.2}$$

where ∇_x is the gradient with respect to the Eulerian variable x, $\cdot$ and $:$ represent the inner products of vectors and matrices respectively.

Since the domain Ω_φ is unknown, in order to solve (2.2) it will be necessary to rewrite the balance equations over the well known reference configuration. This can be done through the change of variables

$$(x_1, x_2, x_3) = \varphi(X_1, X_2, X_3).$$

Consequently, we obtain the following formulation

$$-\int_{\Omega_R} \boldsymbol{T} : \nabla_X \boldsymbol{v} \, dX + \int_{\Omega_R} \boldsymbol{f}_R \cdot \boldsymbol{v} \, dX + \int_{\Gamma_{R1}} \boldsymbol{g}_R \cdot \boldsymbol{v} \, ds = 0, \ \forall \boldsymbol{v} : \overline{\Omega}_R \mapsto \mathbb{R}^3, \ \boldsymbol{v}_{|\Gamma_{R0}} = 0, \tag{2.3}$$

where

$$\boldsymbol{T}(X) = (\det \boldsymbol{F}(X)) \boldsymbol{\sigma}(\varphi(X)) \boldsymbol{F}^{-T}(X), \qquad \boldsymbol{f}_R(X) = (\det \boldsymbol{F}(X)) \boldsymbol{f}_\varphi(\varphi(X)), \tag{2.4a}$$

$$\boldsymbol{g}_R(X) = \boldsymbol{g}_\varphi(\varphi(X)) \frac{ds_\varphi}{ds}, \qquad \boldsymbol{v}(X) = \boldsymbol{v}_\varphi(\varphi(X)). \tag{2.4b}$$

The tensor $\boldsymbol{T}$ is called the first Piola-Kirchhoff stress tensor and $\boldsymbol{F} = \nabla_X \varphi$ is the gradient of deformation. The terms ds and ds_φ represent the surface integration elements in the reference and actual configurations respectively.

For the System (2.1) to be a complete well posed problem, an additional equation specifying the material properties is required. This relation is called the constitutive law and it relates the stress tensor $\boldsymbol{\sigma}$ (or $\boldsymbol{T}$) to the gradient of deformation $\boldsymbol{F}$.

The material is said to be hyperelastic if there exists a functional $W = W(\boldsymbol{F})$ called the strain energy such that $\boldsymbol{T} = \partial W / \partial \boldsymbol{F}$.

The problem of existence and uniqueness of solutions of Systems (2.3)-(2.4) is very difficult as long as the constitutive law is nonlinear and/or anisotropic. Many simplified versions of this problem have been addressed in the literature since the original work of Rivlin [15]. A typical result of existence of solutions of Systems (2.3)-(2.4) is obtained in the particular case where the strain energy function is polyconvex and having a particular growth condition, see for instance [1]. However, in practice these hypotheses are known to be too strong. For instance, existence of solutions in the case of a quasiconvex strain energy function is still an open problem.

3 Geometry of revolution

In the present study, the domain Ω_R is assumed to have a structure of revolution. Namely, we see the outer surface of Ω_R as generated by rotating a two-dimensional meridian closed curve around an axis, for instance the vertical axis, see Fig. 1. Moreover, we assume that the generating curve is from one side of the axis of revolution, however its distance to the axis may be equal to 0.

3.1 Two-dimensional domain

Let $\omega \subset \mathbb{R}^2$ be an open bounded connected domain representing the interior of the generating closed meridian curve in the vertical plane. A point in ω will be identified by its coordinates that we denote by (r,z). Therefore a material point in the reference configuration can be identified by its cylindrical coordinates (r,z,θ) where $(r,z) \in \overline{\omega}$ and $\theta \in [0,2\pi]$. Moreover we have

$$X = (X_1, X_2, X_3) = (r\cos\theta, r\sin\theta, z).$$

Similarly a material point in the actual configuration Ω_φ can be given in terms of the cylindrical coordinates by $\varphi(X(r,z,\theta))$ where $(r,z,\theta) \in \overline{\omega} \times [0,2\pi]$.

Let γ be the boundary of the domain ω. We denote by γ_a the common curve between the meridian curve and the axis of revolution if it exists, otherwise we set

$$\gamma_a = \emptyset.$$

The boundary Γ_R can be obtained by rotation of $\gamma \setminus \gamma_a$ about the axis of revolution. The parts of γ corresponding to Γ_{R0} and Γ_{R1} will be denoted by γ_0 and γ_1 respectively. Consequently, we write

$$\gamma = \overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \overline{\gamma}_a.$$

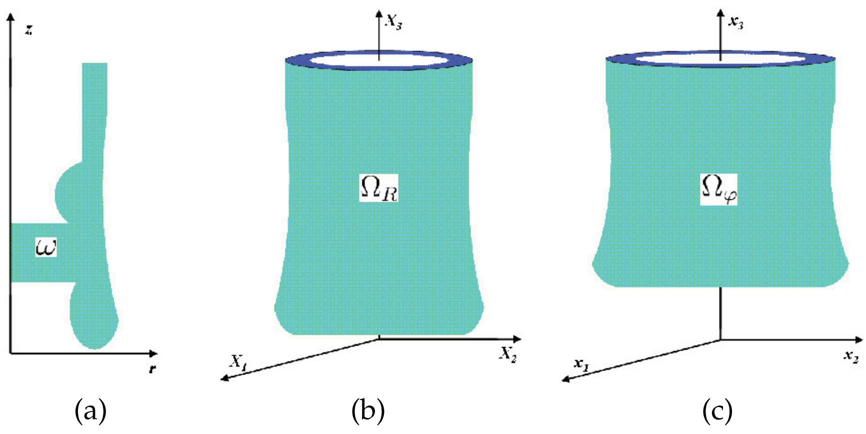


Figure 1: (a): the two-dimensional domain ω. (b): the reference configuration Ω_R. (c): the actual configuration Ω_φ.

3.2 Parametrization of the boundary

In order to correctly represent the boundary terms in the balance equations, a parametrization of γ is essential. Let η be a parametrization of γ of the form:

$$\eta : t \in I = [a,b] \mapsto \eta(t) = (r(t), z(t)) \in \mathbb{R}^2.$$

Since γ is the union of three parts, namely γ_0, γ_1 and γ_a, we can assume that the interval I is the union of three sets

$$I = I_0 \cup I_1 \cup I_a,$$

where $\eta_{|I_0}$ is a parametrization of γ_0, $\eta_{|I_1}$ is a parametrization of γ_1, and $\eta_{|I_a}$ is a parametrization of γ_a.

Let $\psi_R : \overline{\omega} \mapsto \mathbb{R}^3$ defined by

$$\psi_R(r,z) = (r, 0, z),$$

and denote by $\boldsymbol{R}_\theta$ the rotation of angle θ about the axis of revolution. Therefore the mapping

$$\chi_R : (t,\theta) \in I_0 \cup I_1 \times [0, 2\pi] \mapsto \chi_R(t,\theta) \in \mathbb{R}^3,$$

defined by

$$\chi_R(t,\theta) = \boldsymbol{R}_\theta \psi_R(\eta(t)),$$

is a parametrization of Γ_R.

Two tangent vectors (supposed linearly independent) to Γ_R can then be obtained:

$$\boldsymbol{\tau}_1(t,\theta) = \partial_t \chi_R(t,\theta) \quad \text{and} \quad \boldsymbol{\tau}_2(t,\theta) = \partial_\theta \chi_R(t,\theta).$$

Let $\boldsymbol{\Omega}_Z$ be the skew-symmetric linear mapping defined by:

$$\boldsymbol{\Omega}_Z(\boldsymbol{X}) = (0,0,1) \wedge \boldsymbol{X}, \quad \forall \boldsymbol{X} \in \mathbb{R}^3,$$

where $\wedge$ represents the cross product.

The characteristic matrix of $\boldsymbol{\Omega}_Z$ in the Euclidean base will be also denoted by $\boldsymbol{\Omega}_Z$ and is given by

$$\boldsymbol{\Omega}_Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

The following two lemmas are needed later on in some proofs.

Lemma 3.1. *The derivative $\boldsymbol{R}'_\theta$ with respect to θ of the mapping $\theta \mapsto \boldsymbol{R}_\theta$ is given by*

$$\boldsymbol{R}'_\theta = \boldsymbol{R}_\theta \boldsymbol{\Omega}_Z.$$

Lemma 3.2. *The cross product of two vectors is invariant by rotation. More precisely, for every vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ of $\mathbb{R}^3$, we have:*

$$\boldsymbol{R}_\theta(\boldsymbol{u} \wedge \boldsymbol{v}) = \boldsymbol{R}_\theta \boldsymbol{u} \wedge \boldsymbol{R}_\theta \boldsymbol{v}.$$

Proposition 3.1. *The surface integration element in the reference configuration is given by*

$$ds_R = J_R(t)\, dt\, d\theta, \tag{3.2}$$

where $J_R(t) = ||\boldsymbol{\tau}_1(t,\theta) \wedge \boldsymbol{\tau}_2(t,\theta)||$ does not depend on θ.

Proof. We have

$$\boldsymbol{\tau}_1(t,\theta) = \boldsymbol{R}_\theta \nabla \psi_R(\eta(t))\dot{\eta}(t) \quad \text{and} \quad \boldsymbol{\tau}_2(t,\theta) = \boldsymbol{R}_\theta \boldsymbol{\Omega}_Z \psi_R(\eta(t)), \tag{3.3}$$

therefore

$$\boldsymbol{\tau}_1(t,\theta) \wedge \boldsymbol{\tau}_2(t,\theta) = (\boldsymbol{R}_\theta \nabla \psi_R(\eta(t))\dot{\eta}(t)) \wedge (\boldsymbol{R}_\theta \boldsymbol{\Omega}_Z \psi_R(\eta(t))). \tag{3.4}$$

Due to Lemma (3.2) we get

$$\boldsymbol{\tau}_1(t,\theta) \wedge \boldsymbol{\tau}_2(t,\theta) = \boldsymbol{R}_\theta[\nabla \psi_R(\eta(t))\dot{\eta}(t) \wedge \boldsymbol{\Omega}_Z \psi_R(\eta(t))] = \boldsymbol{R}_\theta \boldsymbol{J}_R(t), \tag{3.5}$$

where we set

$$\boldsymbol{J}_R(t) = \nabla \psi_R(\eta(t))\dot{\eta}(t) \wedge \boldsymbol{\Omega}_Z \psi_R(\eta(t)). \tag{3.6}$$

Since $\boldsymbol{R}_\theta$ is a rotation matrix, then $J_R(t) = ||\boldsymbol{J}_R(t))||$ does not depend on θ. □

4 Axisymmetric mechanical model

4.1 Axisymmetric assumptions

In order to obtain an axisymmetric model, some assumptions must be made. The assumptions concerned with the deformation, the boundary conditions, the external forces and the constitutive law. Indeed, we assume that after deformation, the domain keeps a geometry of revolution. To this end, the external forces, the boundary conditions and the constitutive law must be invariant by rotation about the axis of revolution.

4.1.1 Deformation

In our study, we are interested only in deformations which preserve the property of axisymmetry, in other words, those which can be deduced by rotation about the vertical axis of the three-dimensional deformation of a meridian curve. Then the deformation φ we are interested in satisfies:

$$\varphi(X(r,z,\theta)) = \boldsymbol{R}_\theta \psi(r,z) \in \mathbb{R}^3, \quad \forall (r,z,\theta) \in \bar{\omega} \times [0,2\pi],$$

where

$$\psi : (r,z) \in \bar{\omega} \mapsto \psi(r,z) \in \mathbb{R}^3,$$

is a one-to-one mapping and sufficiently smooth. Consequently, the main unknown of our problem becomes the mapping ψ instead of φ.

For the readers who are interested in the development of the axisymmetric model in a rigorous functional framework, we refer to the work of Bernardi et al. [2]. In the present study we aim at developing the axisymmetric model in a formal way without making rigorous attention to the functional spaces where we are seeking for a solution. Indeed, the functional spaces where the unknowns φ and ψ live are related to the choice of the constitutive law. For instance, if we consider the Saint-Venant Kirchhoff constitutive law, the unknown φ will be sought in $(W^{1,4}(\Omega_R))^3$, then by the embedding of $(W^{1,4}(\Omega_R))^3$ in $(C(\Omega_R))^3$ we deduce that φ is continuous, and also ψ is continuous.

It is worth mentioning that if the function φ belongs to some Sobolev space, the function ψ will belong to a weighted Sobolev space with weights of the form r^α for some $-1 \le \alpha \le 1$ and where r is the distance to the axis of revolution. Moreover, it is essential to notice that ψ vanishes on the boundary γ_a where the trace is defined in a weak sense, see Bernardi et al. [2].

In our modeling, we will always make assumptions that imply the continuity of φ and consequently that of ψ. A simple verification leads, in the case where $\gamma_a \neq \emptyset$, to the fact that ψ is defined and equal to zero on γ_a. Indeed, the Cartesian coordinates are expressed in terms of the cylindrical coordinates by the relation $X(r,z,\theta) = (r\cos\theta, r\sin\theta, z)$. At every point of Ω_R which is not on the axis of revolution, the angle θ is uniquely determined up to 2π. However on the axis of revolution, every point can be determined by all values of the angle θ. Since the deformation φ is considered to be continuous and that it is of the form $\varphi = \boldsymbol{R}_\theta\psi$, and since on the axis of revolution

$$X_1 = X_2 = r = 0,$$

then on γ_a we have, for every value $\theta \in [0, 2\pi]$

$$\varphi(0,0,z) = \boldsymbol{R}_\theta\psi(0,z) = \big(\cos\theta\psi_1 - \sin\theta\psi_2, \sin\theta\psi_1 + \cos\theta\psi_2, \psi_3\big). \tag{4.1}$$

This implies

$$\psi_1(0,z) = \psi_2(0,z) = \varphi_1(0,0,z) = \varphi_2(0,0,z) = 0.$$

Here we have

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) \quad \text{and} \quad \psi = (\psi_1, \psi_2, \psi_3).$$

4.1.2 Applied forces

The actual configuration Ω_φ is subjected to two types of applied forces: body forces $\boldsymbol{f}_\varphi$ and surface forces $\boldsymbol{g}_\varphi$. In the axisymmetric model, these forces must be considered invariant by rotation about the axis of revolution. Therefore, we assume that there exist mappings $\boldsymbol{f} : \omega \mapsto \mathbb{R}^3$ and $\boldsymbol{g} : \gamma \mapsto \mathbb{R}^3$, such that

$$\forall (r,z,\theta) \in \omega \times [0,2\pi], \quad \boldsymbol{f}_\varphi(x(r,z,\theta)) = \boldsymbol{R}_\theta \boldsymbol{f}(r,z), \tag{4.2a}$$

and

$$\forall (r,z,\theta) \in \gamma \times [0,2\pi], \quad \boldsymbol{g}_\varphi(x(r,z,\theta)) = \boldsymbol{R}_\theta \boldsymbol{g}(r,z). \tag{4.2b}$$

4.1.3 Dirichlet boundary condition

The boundary Γ_{R0} is subjected to a Dirichlet boundary condition, namely we impose the following condition

$$\varphi = \varphi_0, \quad \text{on } \Gamma_{R0}, \quad \text{where } \varphi_0 : \overline{\Omega}_R \mapsto \mathbb{R}^3. \tag{4.3}$$

In order to be consistent with the axisymmetry assumption, we suppose that φ_0 is invariant by rotation. Therefore, there exists a mapping

$$\psi_0 : \bar{\omega} \mapsto \mathbb{R}^3,$$

such that

$$\varphi_0(X(r,z,\theta)) = \boldsymbol{R}_\theta \psi_0(r,z), \quad \forall (r,z,\theta) \in \overline{\omega} \times [0,2\pi]. \tag{4.4}$$

Consequently, the Dirichlet boundary condition on φ leads to the following Dirichlet boundary condition on ψ:

$$\psi = \psi_0, \quad \text{on } \gamma_0. \tag{4.5}$$

Let us introduce the following two spaces

$$\mathbb{W}_0 = \{\psi = (\psi_1, \psi_2, \psi_3) : \bar{\omega} \mapsto \mathbb{R}^3, \ \psi = 0, \ \text{on } \gamma_0 \ \text{ and } \ \psi_1 = \psi_2 = 0, \ \text{on } \gamma_a\}, \tag{4.6a}$$

$$\mathbb{W} = \{\psi = (\psi_1, \psi_2, \psi_3) : \bar{\omega} \mapsto \mathbb{R}^3, \ \psi = \psi_0, \ \text{on } \gamma_0 \ \text{ and } \ \psi_1 = \psi_2 = 0, \ \text{on } \gamma_a\}. \tag{4.6b}$$

4.1.4 Constitutive law

Similarly, for the axisymmetric model to be valid, the constitutive law must satisfy the following conditions:

1. The law is independent explicitly from θ.
2. The law satisfies the principle of frame indifference:

$$\boldsymbol{T}(\boldsymbol{R}_\theta \boldsymbol{F}) = \boldsymbol{R}_\theta \boldsymbol{T}(\boldsymbol{F}). \tag{4.7}$$

3. The rotations about the axis of revolution belong to the symmetry group of the constitutive law:

$$\boldsymbol{T}(\boldsymbol{F}\boldsymbol{R}_\theta) = \boldsymbol{T}(\boldsymbol{F})\boldsymbol{R}_\theta. \tag{4.8}$$

4.2 Balance equations in cylindrical coordinates

Similar to what we have done for the parametrization of the boundary Γ_R, the mapping

$$\chi : (t,\theta) \in I_0 \cup I_1 \times [0,2\pi] \mapsto \chi(t) \in \mathbb{R}^3, \tag{4.9}$$

defined by $\chi(t,\theta) = \boldsymbol{R}_\theta \psi(\eta(t))$ is a parametrization of Γ. Moreover, the surface integration element in the actual configuration is given by

$$ds_\varphi = J(t)\, dtd\theta, \quad \text{where } \; J(t) = ||\nabla\psi(\eta(t))\dot{\eta}(t) \wedge \boldsymbol{\Omega}_Z \psi(\eta(t))||. \tag{4.10}$$

Let $\boldsymbol{g}_i = \partial_i \boldsymbol{x}$, $i = r,z,\theta$, be the vectors partial derivatives of $\varphi(X(r,z,\theta))$ with respect to r,z and θ respectively. Moreover, let $\boldsymbol{b}_i = \partial_i \psi$, $i = r,z$ and $\boldsymbol{b}_\theta = \boldsymbol{\Omega}_Z \psi$. Similarly, we define

$$\boldsymbol{g}_{Ri} = \partial_i \boldsymbol{R}_\theta \psi_R, \quad i = r,z,\theta \quad \text{and} \quad \boldsymbol{b}_{Ri} = \partial_i \psi_R, \quad i = r,z \quad \text{and} \quad \boldsymbol{b}_{R\theta} = \boldsymbol{\Omega}_Z \psi_R.$$

We denote by b_R the determinant of the matrix whose columns are the vectors $\boldsymbol{b}_{Ri}$, $i = r,z,\theta$.

Proposition 4.1. *For $i = r,z,\theta$, $\boldsymbol{g}_i$ is given in terms of $\boldsymbol{b}_i$ by $\boldsymbol{g}_i = \boldsymbol{R}_\theta \boldsymbol{b}_i$. Moreover, if we denote by $\boldsymbol{g}^i$ the dual basis of $\boldsymbol{g}_i$, then*

$$\boldsymbol{g}^i = \boldsymbol{R}_\theta \boldsymbol{b}^i,$$

where $\boldsymbol{b}^i$ denotes the dual basis of $\boldsymbol{b}_i$. Similar results are also valid for $\boldsymbol{g}_{Ri}$.

Proof. It can be easily shown that for $i = r,z$, we have $\boldsymbol{g}_i = \boldsymbol{R}_\theta \boldsymbol{b}_i$, the one in θ is a direct consequence from Lemma 3.1. Let us check that $\boldsymbol{g}^i = \boldsymbol{R}_\theta \boldsymbol{b}^i$, $i = r,z,\theta$, represent the dual basis for $\boldsymbol{g}_i$. Indeed, for every i,j we have

$$\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \boldsymbol{R}_\theta \boldsymbol{b}^i \cdot \boldsymbol{R}_\theta \boldsymbol{b}_j = \boldsymbol{R}_\theta^T \boldsymbol{R}_\theta \boldsymbol{b}^i \cdot \boldsymbol{b}_j = \boldsymbol{b}^i \cdot \boldsymbol{b}_j = \delta^i_j. \tag{4.11}$$

So, the proof is completed. $\square$

Let m be the absolute value of the mixed product of the three vectors $\boldsymbol{g}_r$, $\boldsymbol{g}_z$ and $\boldsymbol{g}_\theta$. Therefore,

$$m = |\boldsymbol{g}_r, \boldsymbol{g}_z, \boldsymbol{g}_\theta| = |\boldsymbol{R}_\theta \boldsymbol{b}_r, \boldsymbol{R}_\theta \boldsymbol{b}_z, \boldsymbol{R}_\theta \boldsymbol{b}_\theta| = |\boldsymbol{b}_r, \boldsymbol{b}_z, \boldsymbol{b}_\theta|, \tag{4.12}$$

since $\boldsymbol{R}_\theta$ is a rotation. It is important to notice that m is independent from θ.

Lemma 4.1. *Let $\boldsymbol{M}$ be a matrix of order n and, $\boldsymbol{a}$ and $\boldsymbol{b}$ two vectors of $\mathbb{R}^n$, then*

$$(\boldsymbol{M}\boldsymbol{a}) \otimes (\boldsymbol{M}\boldsymbol{b}) = \boldsymbol{M}(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{M}^T. \tag{4.13}$$

Proposition 4.2. *The gradient of deformation $\boldsymbol{F}$ satisfies*

$$\boldsymbol{F} = \boldsymbol{R}_\theta \Big(\sum_{i=r,z,\theta} \boldsymbol{b}_i \otimes \boldsymbol{b}^i_R \Big) \boldsymbol{R}_\theta^T. \tag{4.14}$$

Proof. First of all, we have

$$\boldsymbol{g}_i = \boldsymbol{F}\boldsymbol{g}_{Ri}, \quad \forall i = r, z, \theta. \tag{4.15}$$

Then

$$\boldsymbol{F} = \sum_{i=r,z,\theta} \boldsymbol{g}_i \otimes \boldsymbol{g}_R^i. \tag{4.16}$$

Using Lemma (4.1) with the fact that $\boldsymbol{g}_i = \boldsymbol{R}_\theta \boldsymbol{b}_i$ and $\boldsymbol{g}_R^i = \boldsymbol{R}_\theta \boldsymbol{b}_R^i$, we get Eq. (4.14). Moreover, using Eq. (4.12) we have

$$m(r,z) = det(\boldsymbol{F}) b_R.$$

So, we complete the proof. $\square$

In cylindrical coordinates let $\boldsymbol{S}_0^i$, for $i = r, z, \theta$, be the stress vectors defined by

$$\boldsymbol{S}_0^i(r,z,\theta) = m(r,z)\boldsymbol{\sigma}(\boldsymbol{x}(r,z,\theta))\boldsymbol{g}^i(r,z,\theta), \quad \forall (r,z,\theta) \in \omega \times [0, 2\pi], \tag{4.17}$$

and let $\boldsymbol{S}^i$, for $i = r, z, \theta$, be the vectors defined by

$$\boldsymbol{S}^i = \boldsymbol{R}_\theta^T \boldsymbol{S}_0^i. \tag{4.18}$$

Substituting $\boldsymbol{\sigma}$ from Eq. (2.4) into Eqs. (4.17) and (4.18) we get

$$\boldsymbol{S}^i = b_R \boldsymbol{R}_\theta^T \boldsymbol{T}(\boldsymbol{F}) \boldsymbol{F}^T \boldsymbol{g}^i. \tag{4.19}$$

The term $\boldsymbol{F}^T\boldsymbol{g}^i$ is actually $\boldsymbol{g}_R^i$, indeed

$$\boldsymbol{F}^T \boldsymbol{g}^i = \Big(\sum_{l=r,z,\theta} \boldsymbol{g}_R^l \otimes \boldsymbol{g}_l \Big) \boldsymbol{g}^i = \sum_{l=r,z,\theta} \boldsymbol{g}_R^l \boldsymbol{g}_l \cdot \boldsymbol{g}^i = \sum_{l=r,z,\theta} \boldsymbol{g}_R^l \delta_l^i = \boldsymbol{g}_R^i. \tag{4.20}$$

Therefore

$$\boldsymbol{S}^i = b_R \boldsymbol{R}_\theta^T \boldsymbol{T}(\boldsymbol{F}) \boldsymbol{g}_R^i. \tag{4.21}$$

Let $\boldsymbol{B}(\psi)$ be the matrix

$$\boldsymbol{B}(\psi) = \sum_{i=r,z,\theta} \boldsymbol{b}_i \otimes \boldsymbol{b}_R^i. \tag{4.22}$$

By Eqs. (4.7) and (4.8) we obtain

$$\boldsymbol{S}^i = b_R \boldsymbol{R}_\theta^T \boldsymbol{T}(\boldsymbol{R}_\theta \boldsymbol{B}(\psi) \boldsymbol{R}_\theta^T) \boldsymbol{R}_\theta \boldsymbol{b}_R^i = b_R \boldsymbol{T}(\boldsymbol{B}(\psi) \boldsymbol{R}_\theta^T) \boldsymbol{R}_\theta \boldsymbol{b}_R^i = b_R \boldsymbol{T}(\boldsymbol{B}(\psi)) \boldsymbol{b}_R^i. \tag{4.23}$$

Theorem 4.1. *Suppose that all of the axisymmetry assumptions mentioned above hold true, then the weak formulation of the system of balance equations in cylindrical coordinates is given by: find $\psi \in \mathbb{W}$, such that*

$$\begin{aligned} &-\int_\omega (\boldsymbol{S}^r \cdot \partial_r \boldsymbol{w} + \boldsymbol{S}^z \cdot \partial_z \boldsymbol{w} + \boldsymbol{S}^\theta \cdot \boldsymbol{\Omega}_Z \boldsymbol{w})\, dr\, dz + \int_\omega \boldsymbol{f} \cdot \boldsymbol{w}\, r\, dr\, dz \\ &\qquad + \int_{I_1} J(t) \boldsymbol{g}_\psi(\eta(t)) \cdot \boldsymbol{w}(\eta(t))\, dt = 0, \quad \forall \boldsymbol{w} \in \mathbb{W}_0. \end{aligned} \tag{4.24}$$

Proof. The weak formulation of the balance equation in the actual configuration is given by

$$\begin{aligned}&-\int_{\Omega_\varphi}\boldsymbol{\sigma}:\nabla_x\boldsymbol{v}_\varphi\,dx+\int_{\Omega_\varphi}\boldsymbol{f}_\varphi\cdot\boldsymbol{v}_\varphi\,dx+\int_{\Gamma_{\varphi1}}\boldsymbol{g}_\varphi\cdot\boldsymbol{v}_\varphi\,ds_\varphi\\&=0,\quad\forall\boldsymbol{v}_\varphi:\overline{\Omega}_\varphi\mapsto\mathbb{R}^3,\quad\boldsymbol{v}_{\varphi_{|\Gamma_{\varphi0}}}=0.\end{aligned}\tag{4.25}$$

The idea is to make the change from Cartesian to cylindrical coordinates in the integrals over Ω_φ. For each $\boldsymbol{v}_\varphi:\Omega_\varphi\to\mathbb{R}^3$, we introduce $\boldsymbol{v}:\bar{\omega}\times[0,2\pi]\mapsto\mathbb{R}^3$ defined by

$$\boldsymbol{v}(r,z,\theta)=\boldsymbol{v}_\varphi(x(r,z,\theta)).$$

We have

$$\partial_i\boldsymbol{v}=\nabla_x\boldsymbol{v}_\varphi\boldsymbol{g}_i,\quad i=r,z,\theta.\tag{4.26}$$

Then

$$\nabla_x\boldsymbol{v}_\varphi=\sum_{i=r,z,\theta}\partial_i\boldsymbol{v}\otimes\boldsymbol{g}^i.\tag{4.27}$$

Consequently

$$\begin{aligned}\boldsymbol{\sigma}:\nabla_x\boldsymbol{v}_\varphi=&\boldsymbol{\sigma}:\sum_{i=r,z,\theta}\partial_i\boldsymbol{v}\otimes\boldsymbol{g}^i=\sum_{i=r,z,\theta}\boldsymbol{\sigma}:\partial_i\boldsymbol{v}\otimes\boldsymbol{g}^i\\=&\sum_{i=r,z,\theta}tr\big(\boldsymbol{\sigma}(\boldsymbol{g}^i\otimes\partial_i\boldsymbol{v})\big)=\sum_{i=r,z,\theta}tr(\boldsymbol{\sigma}\boldsymbol{g}^i\otimes\partial_i\boldsymbol{v}),\end{aligned}\tag{4.28}$$

which yields

$$\boldsymbol{\sigma}:\nabla_x\boldsymbol{v}_\varphi=\sum_{i=r,z,\theta}\boldsymbol{\sigma}\boldsymbol{g}^i\cdot\partial_i\boldsymbol{v}.\tag{4.29}$$

The integrals over Ω_φ in (4.25) become:

$$-\int_{\Omega_\varphi}\boldsymbol{\sigma}:\nabla_x\boldsymbol{v}_\varphi dx=-\int_{\omega\times[0,2\pi]}\sum_{i=r,z,\theta}\boldsymbol{R}_\theta\boldsymbol{S}^i\cdot\partial_i\boldsymbol{v}\,dr\,dz\,d\theta,\tag{4.30a}$$

$$\int_{\Omega_\varphi}\boldsymbol{f}_\varphi\cdot\boldsymbol{v}_\varphi dx=\int_{\omega\times[0,2\pi]}\boldsymbol{R}_\theta\boldsymbol{f}(r,z)\cdot\boldsymbol{v}\,r\,dr\,dz\,d\theta,\tag{4.30b}$$

where the vectors $\boldsymbol{S}^i$, $i=r,z,\theta$, are defined in (4.18).

Let us now consider the term with the surface integral. We use the parametrization of the boundary $\Gamma_{\varphi1}$ given in (4.9) and $ds_\varphi=J(t)\,dt\,d\theta$ where $J(t)$ is given in Eq. (4.10):

$$\int_{\Gamma_{\varphi1}}\boldsymbol{g}_\varphi\cdot\boldsymbol{v}_\varphi\,ds_\varphi=\int_{I_1\times[0,2\pi]}J(t)\boldsymbol{R}_\theta\boldsymbol{g}(\eta(t))\cdot\boldsymbol{v}(\eta(t),\theta)\,dt\,d\theta.\tag{4.31}$$

Then Eq. (4.25) becomes

$$\begin{aligned}&-\int_{\omega\times[0,2\pi]}\sum_{i=r,z,\theta}\boldsymbol{R}_\theta\boldsymbol{S}^i\cdot\partial_i\boldsymbol{v}\,dr\,dz\,d\theta+\int_{\omega\times[0,2\pi]}\boldsymbol{R}_\theta\boldsymbol{f}(r,z)\cdot\boldsymbol{v}\,rdr\,dz\,d\theta\\&+\int_{I_1\times[0,2\pi]}\boldsymbol{R}_\theta J(t)\boldsymbol{g}(\eta(t))\cdot\boldsymbol{v}(\eta(t),\theta)\,dt\,d\theta=0.\end{aligned}\tag{4.32}$$

In order to write the problem over the two-dimensional domain ω we restrict the space of test functions to those of the form

$$\boldsymbol{v}(r,z,\theta)=\boldsymbol{R}_\theta \boldsymbol{w}(r,z), \quad \text{where } \boldsymbol{w}:\bar{\omega}\longrightarrow \mathbb{R}^3, \quad \boldsymbol{w}=0, \quad \text{on } \gamma_0. \tag{4.33}$$

The test functions $\boldsymbol{w}=(w_1,w_2,w_3)$ satisfy $\boldsymbol{w}=0$, on γ_0 and $w_1=w_2=0$, on γ_a. Consequently, Eq. (4.32) becomes, for such test functions,

$$\begin{aligned} &-\int_\omega (\boldsymbol{S}^r\cdot\partial_r\boldsymbol{w}+\boldsymbol{S}^z\cdot\partial_z\boldsymbol{w}+\boldsymbol{S}^\theta\cdot\boldsymbol{\Omega}_Z\boldsymbol{w})\,dr\,dz+\int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,r\,dr\,dz \\ &\quad+\int_{I_1} J(t)\boldsymbol{g}(\eta(t))\cdot\boldsymbol{w}(\eta(t))\,dt=0, \end{aligned} \tag{4.34}$$

which is exactly Eq. (4.24). □

In order to simplify the numerical implementation of the two-dimensional equilibrium equation, it would be better to write it in matrix form.

Let $\boldsymbol{A}(\psi)$ be the matrix

$$\boldsymbol{A}(\psi)=\big(\partial_r\psi\,|\,\partial_z\psi\,|\,\boldsymbol{\Omega}_Z\psi\big),$$

then we have

$$\boldsymbol{B}(\psi)=\boldsymbol{A}(\psi)\boldsymbol{A}(\psi_R)^{-1},$$

where $\boldsymbol{B}(\psi)$ is the matrix introduced in (4.22). Let $\boldsymbol{S}$ be the matrix that has as column vectors the vectors $\boldsymbol{S}^i$, $i=r,z,\theta$. We write

$$\boldsymbol{S}=(\boldsymbol{S}^r\,|\,\boldsymbol{S}^z\,|\,\boldsymbol{S}^\theta).$$

Then we have

$$\boldsymbol{S}=b_R\boldsymbol{T}(\boldsymbol{B}(\psi))\boldsymbol{A}(\psi_R)^{-T}. \tag{4.35}$$

Eq. (4.24) can be rewritten

$$\begin{aligned} &-\int_\omega \boldsymbol{T}(\boldsymbol{B}(\psi))\boldsymbol{A}(\psi_R)^{-T}:\boldsymbol{A}(\boldsymbol{w})\,r\,dr\,dz+\int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,r\,dr\,dz \\ &\quad+\int_{I_1} J(t)\boldsymbol{g}(\eta(t))\cdot\boldsymbol{w}(\eta(t))\,dt=0. \end{aligned} \tag{4.36}$$

Therefore we get the two-dimensional equilibrium equation in the following matrix form

$$-\int_\omega \boldsymbol{T}(\boldsymbol{B}(\psi)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz+\int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,r\,dr\,dz+\int_{I_1} J(t)\boldsymbol{g}(\eta(t))\cdot\boldsymbol{w}(\eta(t))\,dt=0. \tag{4.37}$$

4.3 Pressure boundary conditions

In the particular case where the surface force applied to the boundary is a pressure, we can write

$$\boldsymbol{g}_\varphi(x) = -p_{ext}(x)\boldsymbol{n}_\varphi(x), \quad \forall x \in \Gamma_{\varphi 1}, \tag{4.38}$$

where $\boldsymbol{n}_\varphi$ is the outward unit normal to $\Gamma_{\varphi 1}$, and p_{ext} is a real valued function invariant by rotation about the axis of revolution. In fact, p_{ext} is independent from θ, and for simplicity, we are going to write

$$p_{ext}(x) = p_{ext}(r,z).$$

The following lemma is useful for the proof of the next proposition.

Lemma 4.2. *Let $\boldsymbol{M}$ be a 3×3 matrix and, $\boldsymbol{u}$ and $\boldsymbol{v}$ two vectors in $\mathbb{R}^3$. Then, we have*

$$\boldsymbol{Mu} \wedge \boldsymbol{Mv} = (Cof(\boldsymbol{M}))\boldsymbol{u} \wedge \boldsymbol{v}. \tag{4.39}$$

Here $Cof(\boldsymbol{M})$ represents the matrix of cofactors of $\boldsymbol{M}$.

Theorem 4.2. *The term due to the pressure in the equilibrium equation is given by*

$$\int_{\Gamma_{\varphi 1}} \boldsymbol{g}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = -\int_{\Gamma_{\varphi 1}} p_{ext}\, \boldsymbol{n}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = -2\pi \int_{\gamma_1} p_{ext} Cof(\boldsymbol{B}(\psi))\boldsymbol{N} \cdot \boldsymbol{w}\, r\, ds, \tag{4.40}$$

with

$$\boldsymbol{N} = \begin{pmatrix} n_1 \\ 0 \\ n_2 \end{pmatrix},$$

where $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ *is the outward unit normal to γ_1.*

Proof. The change from Eulerian to Lagrangian variables yields

$$\int_{\Gamma_{\varphi 1}} p_{ext}\, \boldsymbol{n}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = \int_{\Gamma_{R1}} p_{ext}\, Cof(\nabla_X \varphi)\, \boldsymbol{n}_R \cdot \boldsymbol{v}_R \, ds_R, \tag{4.41}$$

where $\boldsymbol{n}_R$ is the outward unit normal to Γ_{R1}. Due to the parametrization already given for Γ_{R1}, we can write

$$ds_R = J_R(t)\, dt\, d\theta \quad \text{and} \quad \boldsymbol{n}_R(t,\theta) = \frac{-1}{J_R(t)}\boldsymbol{\nu}(t,\theta), \quad \text{where } \boldsymbol{\nu} = \boldsymbol{\tau}_1 \wedge \boldsymbol{\tau}_2. \tag{4.42}$$

The vector $\boldsymbol{\nu}(t,\theta)$ is

$$\boldsymbol{\nu}(t,\theta) = \boldsymbol{R}_\theta Cof(\boldsymbol{A}(\psi_R(\eta(t))))\boldsymbol{n}(t), \quad \text{where } \boldsymbol{n}(t) = \begin{pmatrix} n_1(t) \\ n_2(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{z}(t) \\ -\dot{r}(t) \\ 0 \end{pmatrix}. \tag{4.43}$$

Indeed, the vectors $\boldsymbol{\tau}_1(t,\theta)$ and $\boldsymbol{\tau}_2(t,\theta)$ can be explicitly written as follows

$$\boldsymbol{\tau}_1 = \partial_t \chi_R = \boldsymbol{R}_\theta \nabla_X \psi_R \,\dot{\eta} = \boldsymbol{R}_\theta (\partial_r \psi_R \;\partial_z \psi_R) \begin{pmatrix} \dot{r} \\ \dot{z} \end{pmatrix} = \boldsymbol{R}_\theta \boldsymbol{A}(\psi_R) \begin{pmatrix} \dot{r} \\ \dot{z} \\ 0 \end{pmatrix}, \tag{4.44a}$$

$$\boldsymbol{\tau}_2 = \partial_\theta \chi_R = \boldsymbol{R}_\theta \boldsymbol{\Omega}_Z \psi_R = \boldsymbol{R}_\theta \boldsymbol{A}(\psi_R) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.44b}$$

Therefore, using the previous lemma we get

$$\boldsymbol{\nu} = \partial_t \chi_R \wedge \partial_\theta \chi_R = \left[\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R) \begin{pmatrix} \dot{r} \\ \dot{z} \\ 0 \end{pmatrix} \right] \wedge \left[\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = Cof(\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R))\boldsymbol{n}. \tag{4.45}$$

Moreover we have

$$Cof(\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R)) = \boldsymbol{R}_\theta Cof(\boldsymbol{A}(\psi_R)).$$

Indeed

$$\begin{aligned} Cof(\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R)) =& det(\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R))(\boldsymbol{R}_\theta \boldsymbol{A}(\psi_R))^{-T} = det(\boldsymbol{A}(\psi_R))\boldsymbol{R}_\theta^{-T} \boldsymbol{A}(\psi_R)^{-T} \\ =& \boldsymbol{R}_\theta det(\boldsymbol{A}(\psi_R))\boldsymbol{A}(\psi_R)^{-T} = \boldsymbol{R}_\theta Cof(\boldsymbol{A}(\psi_R)). \end{aligned} \tag{4.46}$$

On the other hand, we have

$$\nabla_X \varphi = \boldsymbol{R}_\theta \boldsymbol{B}(\psi) \boldsymbol{R}_\theta^{-1}, \tag{4.47}$$

then

$$\begin{aligned} & -\int_{\Gamma_{R1}} p_{ext}\, Cof(\nabla_X \varphi)\, \boldsymbol{n}_R \cdot \boldsymbol{v}_R \, ds_R \\ =& \int_0^{2\pi} \int_{I_1} p_{ext} Cof(\boldsymbol{R}_\theta \boldsymbol{B}(\psi(\eta(t)))\boldsymbol{R}_\theta^{-1}) \boldsymbol{R}_\theta Cof(\boldsymbol{A}(\psi_R)\boldsymbol{n} \cdot \boldsymbol{R}_\theta \boldsymbol{w}(\eta(t))\, dt\, d\theta \\ =& 2\pi \int_{I_1} p_{ext} Cof(\boldsymbol{B}(\psi(\eta(t))))\, Cof(\boldsymbol{A}(\psi_R(\eta(t))))\boldsymbol{n} \cdot \boldsymbol{w}(\eta(t))\, dt. \end{aligned} \tag{4.48}$$

Writing the integration over γ_1, we obtain

$$\int_{\Gamma_{\varphi 1}} p_{ext}\, \boldsymbol{n}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = -2\pi \int_{\gamma_1} p_{ext} Cof(\boldsymbol{B}(\psi))\, Cof(\boldsymbol{A}(\psi_R)\boldsymbol{n} \cdot \boldsymbol{w}\, ds. \tag{4.49}$$

Since

$$\psi_R(r,z) = (r,0,z),$$

we have

$$\boldsymbol{A}(\psi_R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & r \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{hence} \quad Cof(\boldsymbol{A}(\psi_R)) = \begin{pmatrix} -r & 0 & 0 \\ 0 & 0 & -r \\ 0 & -r & 0 \end{pmatrix}.$$

Therefore

$$Cof(\boldsymbol{A}(\psi_R))\boldsymbol{n} = -r\begin{pmatrix} n_1 \\ 0 \\ n_2 \end{pmatrix} = -r\boldsymbol{N}.$$

Consequently, we obtain

$$\int_{\Gamma_{\varphi 1}} \boldsymbol{g}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = -\int_{\Gamma_{\varphi 1}} p_{ext}\, \boldsymbol{n}_\varphi \cdot \boldsymbol{v}_\varphi \, ds_\varphi = -2\pi \int_{\gamma_1} p_{ext} Cof(\boldsymbol{B}(\psi))\boldsymbol{N} \cdot \boldsymbol{w}\, r\, ds. \tag{4.50}$$

The theorem is thus proved. □

4.4 Boundary condition on the axis of revolution

The weak formulation of the equilibrium obtained over the two-dimensional domain ω is: find $\psi \in \mathbb{W}$, such that for every $\boldsymbol{w} \in \mathbb{W}_0$ we have

$$-\int_\omega \boldsymbol{T}(\boldsymbol{B}(\psi)) : \boldsymbol{B}(\boldsymbol{w})\, r\, dr\, dz + \int_\omega \boldsymbol{f} \cdot \boldsymbol{w}\, r\, dr\, dz - \int_{\gamma_1} p_{ext} Cof(\boldsymbol{B}(\psi))\boldsymbol{N} \cdot \boldsymbol{w}\, r\, ds = 0. \tag{4.51}$$

It is clear that during the derivation of the axisymmetric model, two boundary conditions for the two-dimensional domain have arisen

- Dirichlet boundary condition on γ_0.
- Pressure boundary condition on γ_1.

Since

$$\gamma = \overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \overline{\gamma}_a,$$

it remains to specify what boundary condition must be set for the remaining part γ_a when γ intersects the axis of revolution.

Theorem 4.3. *If*

$$\psi = (\psi_1, \psi_2, \psi_3),$$

is a regular weak solution to (4.51) *and* $\boldsymbol{f} \in L^2(\omega)$, *then* ψ *is a solution to the following strong problem*

$$\begin{cases} -div_c \boldsymbol{T}(\boldsymbol{B}(\psi)) = \boldsymbol{f}, & in\ \ \omega, \\ \boldsymbol{T}(\boldsymbol{B}(\psi))\boldsymbol{N} = -p_{ext} Cof(\boldsymbol{B}(\psi))\boldsymbol{N}, & on\ \ \gamma_1, \\ \psi = \psi_0, & on\ \ \gamma_0, \\ \psi_1 = \psi_2 = 0 \quad and \quad T_{31} = 0, & on\ \ \gamma_a, \end{cases} \tag{4.52}$$

where div_c is the divergence of a matrix with respect to the cylindrical coordinates and it is given by

$$div_c \boldsymbol{T} = \begin{pmatrix} \partial_r T_{11} + \partial_z T_{13} + r^{-1}(T_{11} - T_{22}) \\ \partial_r T_{21} + \partial_z T_{23} + r^{-1}(T_{12} + T_{21}) \\ \partial_r T_{31} + \partial_z T_{33} + r^{-1} T_{31} \end{pmatrix}, \quad where\ \boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{4.53}$$

Proof. We choose in problem (4.51) the test function $\boldsymbol{w}$ in $(\mathcal{D}(\omega))^3$. We obtain

$$\langle r\boldsymbol{T}(\boldsymbol{B}(\psi)), \boldsymbol{B}(\boldsymbol{w})\rangle_{\mathcal{D}^*,\mathcal{D}} = \langle r\boldsymbol{f}, \boldsymbol{w}\rangle_{\mathcal{D}^*,\mathcal{D}}, \tag{4.54}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{D}^*,\mathcal{D}}$ represents the dual product in $(\mathcal{D}(\omega))^3$. This gives

$$-\mathrm{div}_c\boldsymbol{T}(\boldsymbol{B}(\psi)) = \boldsymbol{f}, \quad \text{in } (\mathcal{D}^*(\omega))^3. \tag{4.55}$$

Indeed, we have

$$\begin{aligned}\langle r\boldsymbol{T}, \boldsymbol{B}(\boldsymbol{w})\rangle =&\langle rT_{11}, \partial_r w_1\rangle + \left\langle rT_{12}, \frac{-w_2}{r}\right\rangle + \langle rT_{13}, \partial_z w_1\rangle + \langle rT_{21}, \partial_r w_2\rangle \\ &+ \left\langle rT_{22}, \frac{w_1}{r}\right\rangle + \langle rT_{23}, \partial_z w_2\rangle + \langle rT_{31}, \partial_r w_3\rangle + \langle rT_{33}, \partial_z w_3\rangle,\end{aligned} \tag{4.56}$$

then we obtain

$$\begin{aligned}\langle r\boldsymbol{T}, \boldsymbol{B}(\boldsymbol{w})\rangle =&- \langle T_{11} + r\partial_r T_{11}, w_1\rangle - \langle T_{12}, w_2\rangle - \langle r\partial_z T_{13}, w_1\rangle - \langle T_{21} + r\partial_r T_{21}, w_2\rangle \\ &+ \langle T_{22}, w_1\rangle - \langle r\partial_z T_{23}, w_2\rangle - \langle T_{31} + r\partial_r T_{31}, w_3\rangle - \langle r\partial_z T_{33}, w_3\rangle,\end{aligned} \tag{4.57}$$

which implies

$$\begin{aligned}\langle r\boldsymbol{T}, \boldsymbol{B}(\boldsymbol{w})\rangle =&- \left\langle r\Big(\partial_r T_{11} + \partial_z T_{13} + \frac{1}{r}(T_{11} - T_{22})\Big), w_1\right\rangle - \left\langle r\Big(\partial_r T_{21}\right. \\ &\left.+ \partial_z T_{23} + \frac{1}{r}(T_{12} + T_{21})\Big), w_2\right\rangle - \left\langle r\Big(\partial_r T_{31} + \partial_z T_{33} + \frac{1}{r}T_{31}\Big), w_3\right\rangle.\end{aligned} \tag{4.58}$$

Consequently we get

$$\langle -r\mathrm{div}_c\boldsymbol{T}, \boldsymbol{w}\rangle_{\mathcal{D}^*,\mathcal{D}} = \langle r\boldsymbol{f}, \boldsymbol{w}\rangle_{\mathcal{D}^*,\mathcal{D}}, \tag{4.59}$$

thus Eq. (4.55) is well verified in the sense of distributions. The function $\boldsymbol{f}$ being defined almost everywhere, we get the following partial differential equations satisfied by ψ

$$-\mathrm{div}_c\boldsymbol{T}(\boldsymbol{B}(\psi)) = \boldsymbol{f}, \ \text{ a.e. } (r,z) \in \omega. \tag{4.60}$$

The Dirichlet boundary conditions are satisfied in the weak and strong problems. For ψ to be a strong solution, it remains to show that it satisfies the remaining two Neumann type boundary conditions. To this end, we multiply Eq. (4.60) by $\boldsymbol{w} \in \mathbb{W}_0$ and we integrate over ω. This implies

$$-\int_\omega \mathrm{div}_c\boldsymbol{T}(\boldsymbol{B}(\psi)) \cdot \boldsymbol{w}\, dr\, dz = \int_\omega \boldsymbol{f} \cdot \boldsymbol{w}\, dr\, dz. \tag{4.61}$$

Expanding the inner product in the previous equation, we get

$$\begin{aligned}&-\int_\omega \Big[\partial_r T_{11} w_1 + \partial_z T_{13} w_1 + \frac{1}{r}(T_{11} - T_{22})w_1 + \partial_r T_{21} w_2 + \partial_z T_{23} w_2 \\ &+ \frac{1}{r}(T_{21} + T_{12})w_2 + \partial_r T_{31} w_3 + \partial_z T_{33} w_3 + \frac{1}{r}T_{31} w_3\Big]\, dr\, dz = \int_\omega \boldsymbol{f} \cdot \boldsymbol{w}\, dr\, dz.\end{aligned} \tag{4.62}$$

Under the condition of regularity of ψ, and by Green's formula, we obtain

$$\begin{aligned}&\int_\omega \Big[T_{11}\partial_r w_1 + T_{13}\partial_z w_1 - \frac{1}{r}(T_{11}-T_{22})w_1 + T_{21}\partial_r w_2 + T_{23}\partial_z w_2 + \frac{1}{r}(T_{21}\\ &\quad + T_{12})w_2 + T_{31}\partial_r w_3 + T_{33}\partial_z w_3 + \frac{1}{r}T_{31}w_3\Big]\,dr\,dz - \int_\gamma \Big[T_{11}w_1n_1 + T_{13}w_1n_2\\ &\quad + T_{21}w_2n_1 + T_{23}w_2n_2 + T_{31}w_3n_1 + T_{33}w_3n_2\Big]\,ds = \int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,dr\,dz. \end{aligned}\tag{4.63}$$

Now we split the integral over γ into integrals over γ_0, γ_1 and γ_a and we use the fact that $\boldsymbol{w}=0$ over γ_0 and $w_1=w_2=0$ over γ_a. Moreover, on γ_a the second component of the normal vanishes, i.e., $n_2=0$. This allows us to simplify the previous equation and to write

$$\begin{aligned}&\int_\omega \Big[T_{11}\partial_r w_1 + T_{13}\partial_z w_1 - \frac{1}{r}(T_{11}-T_{22})w_1 + T_{21}\partial_r w_2 + T_{23}\partial_z w_2\\ &\quad + \frac{1}{r}(T_{21}+T_{12})w_2 + T_{31}\partial_r w_3 + T_{33}\partial_z w_3 + \frac{1}{r}T_{31}w_3\Big]\,dr\,dz\\ &\quad - \int_{\gamma_1} \boldsymbol{T}(\boldsymbol{B}(\psi))\boldsymbol{N}\cdot\boldsymbol{w}\,ds - \int_{\gamma_a} T_{31}w_3n_1\,ds = \int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,dr\,dz. \end{aligned}\tag{4.64}$$

On the other hand, substituting in (4.51) $\boldsymbol{w}$ by $\boldsymbol{w}/r$ and using

$$\boldsymbol{B}\Big(\frac{\boldsymbol{w}}{r}\Big) = \begin{pmatrix} \frac{\partial_r w_1}{r} - \frac{w_1}{r^2} & -\frac{w_2}{r^2} & \frac{\partial_z w_1}{r} \\ \frac{\partial_r w_2}{r} - \frac{w_2}{r^2} & \frac{w_1}{r^2} & \frac{\partial_z w_2}{r} \\ \frac{\partial_r w_3}{r} - \frac{w_3}{r^2} & 0 & \frac{\partial_z w_3}{r} \end{pmatrix},\tag{4.65}$$

Eq. (4.51) becomes

$$\begin{aligned}&-\int_\omega \Big[T_{11}\partial_r w_1 - T_{11}\frac{w_1}{r} - T_{12}\frac{w_2}{r} + T_{13}\partial_z w_1 + T_{21}\partial_r w_2 - T_{21}\frac{w_2}{r}\\ &\quad + T_{22}\frac{w_1}{r} + T_{23}\partial_z w_2 + T_{31}\partial_r w_3 - T_{31}\frac{w_3}{r} + T_{33}\partial_z w_3\Big]\,dr\,dz\\ &\quad + \int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,dr\,dz - \int_{\gamma_1} p_{ext}Cof(\boldsymbol{B}(\psi))\boldsymbol{N}\cdot\boldsymbol{w}\,ds = 0. \end{aligned}\tag{4.66}$$

Comparing this last equation with Eq. (4.64) we deduce that

$$\boldsymbol{T}(\boldsymbol{B}(\psi))\boldsymbol{N} = -p_{ext}Cof(\boldsymbol{B}(\psi))\boldsymbol{N}, \quad \text{on } \gamma_1 \quad \text{and} \quad T_{31}=0, \quad \text{on } \gamma_a,\tag{4.67}$$

which achieves the proof. □

4.5 Incompressibility

A material is said to be incompressible if its volume does not change during deformation. This can be obtained by writing a kinematic constraint on the deformation, more precisely, by writing

$$det(\boldsymbol{F}) = 1.\tag{4.68}$$

For a hyperelastic incompressible homogeneous material, the new strain energy has the following form

$$W_{inc}(\boldsymbol{F}) = W(\boldsymbol{F}) + p\big(\det(\boldsymbol{F}) - 1\big), \tag{4.69}$$

where p is a Lagrange multiplier associated to the incompressibility constraint, it is called the hydrostatic pressure. The new first Piola-Kirchhoff stress tensor is also given by

$$\boldsymbol{T}_{inc}(\boldsymbol{F}) = \frac{\partial W_{inc}}{\partial \boldsymbol{F}} = \boldsymbol{T}(\boldsymbol{F}) + p\mathrm{Cof}(\boldsymbol{F}), \quad \text{where } \boldsymbol{T}(\boldsymbol{F}) = \frac{\partial W}{\partial \boldsymbol{F}}. \tag{4.70}$$

Consequently, the corresponding variational problem becomes

$$\begin{cases} \text{Find } (\psi, p) \in \mathbb{W} \times L^2(\omega), \text{ such that} \\ \displaystyle\int_\omega \boldsymbol{T}(\boldsymbol{B}(\psi)) : \boldsymbol{B}(\boldsymbol{w}) r\, dr\, dz + \int_\omega p\mathrm{Cof}(\boldsymbol{B}(\psi)) : \boldsymbol{B}(\boldsymbol{w}) r\, dr\, dz \\ \displaystyle\quad + \int_\omega \boldsymbol{f} \cdot \boldsymbol{w}\, r\, dr\, dz + \int_{\gamma_1} p_{ext}\mathrm{Cof}(\boldsymbol{B}(\psi))\boldsymbol{N} \cdot \boldsymbol{w}\, r\, ds = 0, \quad \forall \boldsymbol{w} \in \mathbb{W}_0, \\ \displaystyle\int_\omega q(\det(\boldsymbol{B}(\psi)) - 1)\, r\, dr\, dz = 0, \quad \forall q \in L^2(\omega). \end{cases} \tag{4.71}$$

The problem of existence and uniqueness of solutions of the proposed axisymmetric model (4.71) is very hard and is not addressed in the present paper. In the sequel, we will focus on solving System (4.71) numerically and validating the result by comparing simulations of the full 3D model and the corresponding reduced axisymmetric model.

5 Linearization and solving technique

In order to solve numerically the nonlinear System (4.71), we use the classic Newton-Raphson method and proceed iteratively to construct the solution. The Newton-Raphson method consists on linearizing System (4.71) with respect to the unknowns ψ and p. Then we start from a suitable choice of the initial value (ψ^0, p^0) and iteratively solve the obtained linear system until its solution converges to a solution of the nonlinear System (4.71).

It is worth mentioning that the nonlinearities of the constitutive law that can be handled by this proposed model are related only to the conditions that are required for the linear system to have a solution and for the Newton-Raphson method to converge. For instance, this is the case when the strain energy function is strictly convex, see Proposition 5.1.

Let us now derive the linear system corresponding to the Newton-Raphson method. Given

$$(\psi^0, p^0) \in \mathbb{W} \times L^2(\omega),$$

we construct the sequences $(\psi^k)_{k\geq 1}$ and $(p^k)_{k\geq 1}$ by solving for ψ^{k+1} and p^{k+1} the following linear problem

$$
\left\{
\begin{aligned}
&\text{Find } (\delta\psi^k,\delta p^k)\in \mathbb{W}_0\times L^2(\omega), \text{ such that}\\
&\int_\omega \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi^k))\boldsymbol{B}(\delta\psi^k):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz \;+\int_\omega p^k\frac{\partial \mathrm{Cof}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi^k))\boldsymbol{B}(\delta\psi^k):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz\\
&\;+\int_\omega \delta p^k\mathrm{Cof}(\boldsymbol{B}(\psi^k)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz \;+\int_{\gamma_1} p_{ext}\frac{\partial \mathrm{Cof}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi^k))\boldsymbol{B}(\delta\psi^k)\boldsymbol{N}\cdot\boldsymbol{w}\,rds\\
&\;+\int_\omega \boldsymbol{T}(\boldsymbol{B}(\psi^k)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz \;+\int_\omega p^k\mathrm{Cof}(B(\psi^k)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz\\
&\;+\int_{\gamma_1} p_{ext}\mathrm{Cof}(\boldsymbol{B}(\psi^k))\boldsymbol{N}\cdot\boldsymbol{w}\,rds+\int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,r\,dr\,dz=0,\quad \forall \boldsymbol{w}\in\mathbb{W}_0,\\
&\int_\omega q\mathrm{Cof}(\boldsymbol{B}(\psi^k)):\boldsymbol{B}(\delta\psi^k)\,r\,dr\,dz+\int_\omega q\big(\det(\boldsymbol{B}(\psi^k))-1\big)\,r\,dr\,dz=0,\quad \forall q\in L^2(\omega),
\end{aligned}
\right.
\tag{5.1}
$$

where

$$\delta\psi^k=\psi^{k+1}-\psi^k \quad \text{and} \quad \delta p^k=p^{k+1}-p^k.$$

In the sequel, we omit the superscript k. Let

$$
\begin{aligned}
a(\boldsymbol{v},\boldsymbol{w}) &= \int_\omega \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi))\boldsymbol{B}(\boldsymbol{v}):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz+\int_{\gamma_1} p_{ext}\frac{\partial \mathrm{Cof}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi))\boldsymbol{B}(\boldsymbol{v})\boldsymbol{N}\cdot\boldsymbol{w}\,rds\\
&\quad +\int_\omega p\frac{\partial \mathrm{Cof}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi))\boldsymbol{B}(\boldsymbol{v}):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz, &(5.2a)\\
b(\boldsymbol{w},q) &= \int_\omega q\mathrm{Cof}(\boldsymbol{B}(\psi)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz, &(5.2b)\\
l(\boldsymbol{w}) &= -\int_\omega \boldsymbol{T}(\boldsymbol{B}(\psi)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz-\int_{\gamma_1} p_{ext}\mathrm{Cof}(\boldsymbol{B}(\psi))\boldsymbol{N}\cdot\boldsymbol{w}\,rds\\
&\quad -\int_\omega p\mathrm{Cof}(\boldsymbol{B}(\psi)):\boldsymbol{B}(\boldsymbol{w})\,r\,dr\,dz-\int_\omega \boldsymbol{f}\cdot\boldsymbol{w}\,r\,dr\,dz, &(5.2c)\\
j(q) &= -\int_\omega q(\det(\boldsymbol{B}(\psi))-1)\,r\,dr\,dz. &(5.2d)
\end{aligned}
$$

For each iteration, the linear problem to solve is

$$
\left\{
\begin{aligned}
&\text{Find } (\delta\psi,\delta p)\in\mathbb{W}_0\times L^2(\omega), \text{ such that}\\
&a(\delta\psi,\boldsymbol{w})+b(\boldsymbol{w},\delta p)=l(\boldsymbol{w}),\quad \forall\boldsymbol{w}\in\mathbb{W}_0,\\
&b(\delta\psi,q)=j(q),\quad \forall q\in L^2(\omega),
\end{aligned}
\right.
\tag{5.3}
$$

where

$$a:\mathbb{W}_0\times\mathbb{W}_0\mapsto\mathbb{R} \quad \text{and} \quad b:\mathbb{W}_0\times L^2(\omega)\mapsto\mathbb{R},$$

are bilinear forms and $l\in\mathbb{W}_0^*$ and

$$j\in(L^2(\omega))^*\equiv L^2(\omega).$$

This problem is called a saddle point problem.

Remark 5.1. The bilinear form a is not symmetric in general because of the term due to the external pressure.

We recall the following theorem about existence and uniqueness of solutions to problem (5.3). For more details, we refer to Ern and Guermond [5]. Let

$$\mathcal{N}(b) = \Big\{ \boldsymbol{w} \in \mathbb{W}_0;\ \ \forall q \in L^2(\omega),\ \ b(\boldsymbol{w}, q) = 0 \Big\}.$$

Theorem 5.1. *Problem* (5.3) *admits a unique solution if and only if*

$$\begin{cases} \exists \alpha > 0,\ \inf\limits_{\boldsymbol{v} \in \mathcal{N}(b)} \sup\limits_{\boldsymbol{w} \in \mathcal{N}(b)} \dfrac{a(\boldsymbol{v}, \boldsymbol{w})}{||\boldsymbol{v}||\,||\boldsymbol{w}||} \geq \alpha, \\ \forall \boldsymbol{w} \in \mathcal{N}(b),\ \big[a(\boldsymbol{v}, \boldsymbol{w}) = 0,\ \forall \boldsymbol{v} \in \mathcal{N}(b)\big] \Rightarrow \big[\boldsymbol{w} = 0\big], \\ \exists \beta > 0,\ \ \inf\limits_{q \in L^2(\omega)} \sup\limits_{\boldsymbol{w} \in \mathbb{W}_0} \dfrac{b(\boldsymbol{w}, q)}{||\boldsymbol{w}||\,||q||} \geq \beta. \end{cases} \tag{5.4}$$

Under these conditions, we have the following estimates

$$||\delta\psi|| \leq c_1 ||l|| + c_2 ||j||, \qquad ||\delta p|| \leq c_2 ||l|| + c_3 ||j||, \tag{5.5}$$

where

$$c_1 = \frac{1}{\alpha}, \quad c_2 = \frac{1}{\beta}\Big(1 + \frac{||a||}{\alpha}\Big) \quad \textit{and} \quad c_3 = \frac{||a||}{\beta^2}\Big(1 + \frac{||a||}{\alpha}\Big).$$

Remark 5.2. If the bilinear form a is coercive on $\mathcal{N}(b)$ or on $\mathbb{W}_0$, then the first two conditions on a in (5.4) are satisfied.

Proposition 5.1. *Assume that*

(H_1) *The strain energy function W is strictly convex.*

(H_2) *There is no pressure boundary condition.*

(H_3) *The bounded domain ω is C^1-regular and does not cross the z-axis.*

Then the bilinear form a given in Eq. (5.2a) *is coercive in $\mathcal{N}(b)$. Consequently, System* (5.3) *or equivalently System* (5.1) *has a unique solution.*

Proof. First of all, by assumption (H_2) the third term in the expression of the bilinear form a given in Eq. (5.2a), is actually null. Moreover, for $\boldsymbol{w} \in \mathcal{N}(b)$, we have $b(\boldsymbol{w}, q) = 0$, for every $q \in L^2(\omega)$ and this will be valid also for every ψ. Hence, by differentiating $b(\boldsymbol{w}, q)$ with respect to ψ, it appears that the third term in the expression of the bilinear form a is also null. Therefore, a reduces to

$$a(\boldsymbol{v}, \boldsymbol{w}) = \int_\omega \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{F}}(\boldsymbol{B}(\psi))\boldsymbol{B}(\boldsymbol{v}) : \boldsymbol{B}(\boldsymbol{w})\, r\, dr\, dz.$$

On the other hand, the first Piola-Kirchhoff stress tensor $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \frac{\partial W}{\partial \boldsymbol{F}},$$

hence

$$\frac{\partial T}{\partial \boldsymbol{F}} = \frac{\partial^2 W}{\partial \boldsymbol{F}^2}.$$

By Assumption (H_1), the Hessian matrix $\partial^2 W/\partial \boldsymbol{F}^2$ is definite positive, consequently there exists $\lambda_0 > 0$ (the smallest eigenvalue of the Hessian), such that

$$\frac{\partial T}{\partial \boldsymbol{F}} \boldsymbol{B}(\boldsymbol{w}) : \boldsymbol{B}(\boldsymbol{w}) = \frac{\partial^2 W}{\partial \boldsymbol{F}^2} \boldsymbol{B}(\boldsymbol{w}) : \boldsymbol{B}(\boldsymbol{w}) \geq \alpha \lambda_0 \|\boldsymbol{B}(\boldsymbol{w})\|^2_{L^2(\omega)},$$

for every $\boldsymbol{w} \in \mathbb{W}_0$, where by Assumption (H_3), $\alpha > 0$ represents the distance from ω to the z-axis. From a classic result in Sobolev spaces, since the domain ω is bounded, regular and does not cross the z-axis, the L^2-norm $\|\boldsymbol{B}(\boldsymbol{w})\|_{L^2(\omega)}$ is equivalent to the norm of $\boldsymbol{w}$ in $\mathbb{W}_0$ where here we suppose that the space $\mathbb{W}_0$ is some appropriate Sobolev space satisfying the boundary conditions already considered in the definition of $\mathbb{W}_0$ given in Eq. (4.6a). Therefore the bilinear form a is coercive in $\mathcal{N}(b) \subset \mathbb{W}_0$.

It can be easily verified that the bilinear form b given in Eq. (5.2b) satisfies the third condition of Theorem (5.1). Consequently, since the three conditions of Theorem (5.1) are satisfied, then System (5.3) or equivalently System (5.1) has a unique solution. $\square$

6 Numerical simulations

6.1 Axisymmetric model for the left ventricle deformation

In this section we apply the axisymmetric model to the deformation of the left ventricle in the heart. It is well known in cardiology that the left ventricle has approximately the shape of a body of revolution. Its external surface can be considered as a half of an ellipsoid. The 3D geometry representing the left ventricle and the corresponding domain ω are represented on Fig. 2.

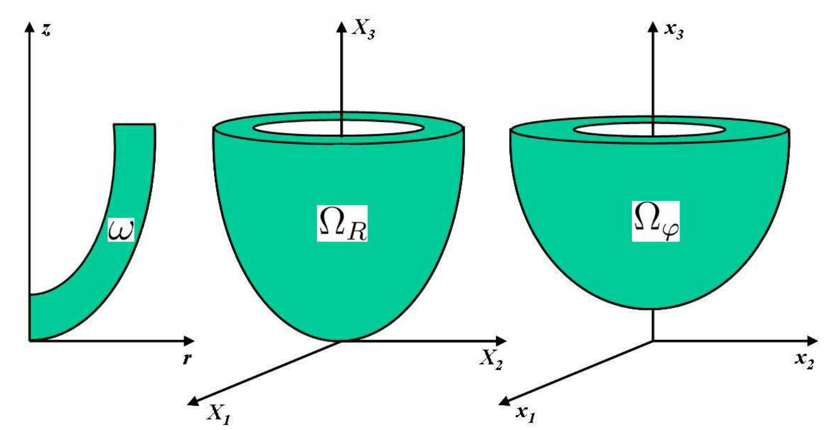
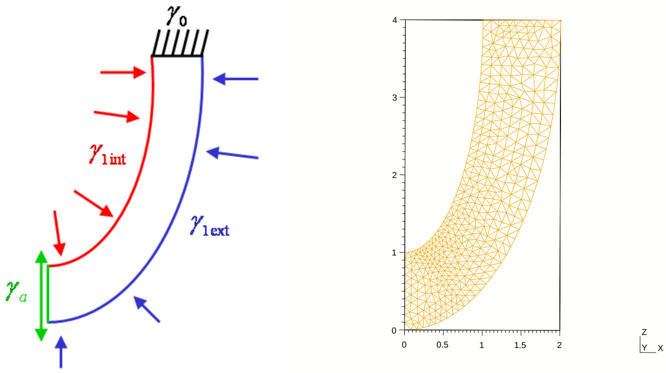


Figure 2: Axisymmetric model for the left ventricle. Left: the two-dimensional domain ω. Center: the reference configuration Ω_R. Right: the actual configuration Ω_φ.



(a) (b)

Figure 3: (a): the domain ω with the repartition of the boundary conditions. (b): the 2D mesh of the domain ω.

In this example, we neglect the volume forces which in general correspond to gravity forces, however as surface forces we impose a nonzero uniform pressure on the internal boundary γ_{1int} that corresponds to the left ventricle cavity (endocardium) and a zero pressure on the external boundary γ_{1ext} that corresponds to the external surface of the left ventricle (epicardium). The internal pressure models the blood pressure on the endocardium. Moreover, we impose no displacement on the boundary γ_0 which corresponds to the base of the left ventricle. Fig. 3 illustrates the repartition of the boundary conditions on the boundary γ of the 2D meridian domain ω.

Several experimental constitutive laws have been proposed for the cardiac tissue. In this paper we consider the constitutive law proposed by Lin and Yin Lin and Yin [13]. This constitutive law is hyperelastic, incompressible, homogeneous and transversely isotropic because of the existence of a preferred orientation, denoted by $\boldsymbol{\tau}$, of the cardiac fibers that constitute the muscle. The corresponding strain energy is given in terms of the invariants I_1 and I_4 by:

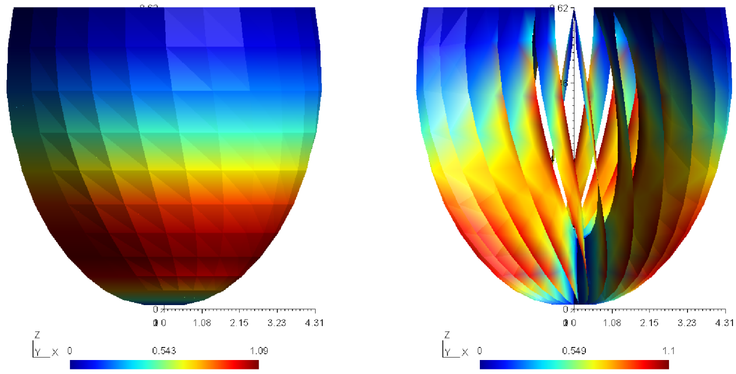
$$W(I_1, I_4) = C_1(e^Q - 1), \quad \text{with } Q = C_2(I_1-3)^2 + C_3(I_1-3)(I_4-1) + C_4(I_4-1)^2, \tag{6.1}$$

where

$$I_1 = tr(\boldsymbol{F}^T\boldsymbol{F}) \quad \text{and} \quad I_4 = \|\boldsymbol{F}\boldsymbol{\tau}\|^2.$$

As for the cardiac fibers, Streeter [18] has conjectured that the left ventricle fibers run as geodesics on a nested set of toroidal surfaces and this conjecture has been checked by Mourad et al. [14] using experimental data on the cardiac fiber orientation. In this section we considered a left ventricle geometrical model with a realistic fiber structure corresponding to helical fibers whose directions change gradually from about 60° on the epicardium to about 0° at the middle of the thickness (almost circular fibers), and to about -60° on the endocardium. This structure is consistent with what has been suggested in [18] and [14].

All of the numerical simulations have been performed using the finite element software Freefem++ [11], and we used the software Gmsh [9] for the visualization. The



(a) (b)

Figure 4: (a): the displacement of the full three-dimensional model. (b): the displacement of the axisymmetric model where the meridian curve has been rotated 24 times for visualization.

numerical simulations of the axisymmetric model have shown the 3D deformation of the 2D meridian domain (the twisting phenomena), see Fig. 4. The boundary γ_a remains on the axis of revolution and just it moves up or down on that axis. Due to the Dirichlet boundary condition, the boundaryγ_0 remains fixed.

By incrementing the value of the internal pressure modeling the filling of the left ventricle cavity by the blood, we noticed the increase in the cavity volume of the left ventricle. On the other hand, when decreasing the value of the internal pressure modeling the blood ejection phase we noticed the twist in the meridian domain, the increase of the left ventricle thickness, the shortening of the left ventricle big axis with a decrease in the cavity volume which are all consistent with experimental observations of the cardiac contraction, see for instance Fung [6], Robb and Robb [16], Streeter [18]. These observations are mainly due to the helical fiber structure in the heart. This shows how well the proposed axisymmetric model can reproduce the experimental observations.

6.2 Comparison to full 3D

In order to compare the numerical results of both the full 3D and the axisymmetric models, the 3D mesh has been generated from the 2D mesh by rotation about the axis of revolution. This has been done using the 3D mesh generator in the software Freefem++ [11].

In the comparison, we considered two different meshes: the first mesh is a non-refined mesh (Table 1) and the second mesh is a refined mesh (Table 2). In the following tables, we summarize the properties of the 2D and 3D meshes. We denote by Nb_{elem} the number of elements (triangles in 2D, tetrahedra in 3D) in the mesh, Nb_{node} the number of nodes, DoF_{ψ} the degrees of freedom for the deformation ψ, DoF_p the degrees of freedom for the hydrostatic pressure p, and Nb_{sys} the total number of degrees of freedom (the total number of unknowns) in the final linear system to solve

Table 1: Comparison between the 2D and 3D meshes.

	Nb_{elem}	Nb_{node}	DoF_ψ	DoF_p	Nb_{sys}	t
2D mesh	70	51	171	51	564	22
3D mesh	5000	1179	8007	1179	25200	1220

Table 2: Comparison between the 2D and 3D refined meshes.

	Nb_{elem}	Nb_{node}	DoF_ψ	DoF_p	Nb_{sys}	t
2D mesh	198	125	447	125	1466	60
3D mesh	14450	2981	21511	2981	67514	∞

(Nb_{sys} = 3*DoF_ψ+DoF_p). We denote by t the wall time in seconds that takes to solve serially the linearized system on an Intel Core (TM)2 CPU T5600-1.83GHz.

The 3D mesh of the full 3D domain has been generated by rotation of the 2D mesh of a closed meridian curve using a particular tool of the software Freefem3d++ [11]. This makes easy the comparison of the results between the full 3D model and the axisymmetric model at individual points. The comparison for the whole mesh is obtained by computing the discrete L^∞ and L^2 norms of the absolute and the relative errors of displacements at all points of one deformed meridian curve. For instance, for the mesh of Table 1 that contains 51 nodes, for an external pressure $p_{ext} = 8$, the L^∞ norm of the relative error of displacements is 1% and the L^2 norm of the relative error is 0.72%. In Table 3, we summarize the numerical results of both the L^∞ and L^2 norms of both the absolute and the relative errors for five different choices of the external pressure p_{ext}.

The numerical results show very similar deformation, the modulus of the deformations in both models are very close together. This observation can also be seen on Fig. 4. On the other hand, we noticed the huge difference in the computational cost. In the case of the non-refined mesh, the axisymmetric model takes about 22 seconds to converge, whereas the full 3D problem takes more than 20 minutes to converge. When refining the mesh, the full 3D problem did not converge at all whereas the axisymmetric model does converge with no big changes in the solution compared to the result obtained with the non-refined mesh. This last remark can be related to the fact that the non-refined mesh contains enough elements so that the corresponding numerical solution is very close to the exact solution.

Table 3: The discrete L^∞ and L^2 norms of the absolute and the relative errors of displacements that correspond to five different values of the external pressure p_{ext} (results corresponding to the mesh that contains 51 nodes given in Table 1).

p_{ext}	Absolute L^∞	Relative L^∞	Absolute L^2	Relative L^2
8	$8.6\ 10^{-3}$	$1.0\ 10^{-2}$	$2.9\ 10^{-3}$	$7.2\ 10^{-3}$
10	$1.0\ 10^{-2}$	$1.2\ 10^{-2}$	$3.6\ 10^{-3}$	$8.1\ 10^{-3}$
13	$1.3\ 10^{-2}$	$1.4\ 10^{-2}$	$4.8\ 10^{-3}$	$8.9\ 10^{-3}$
15	$1.5\ 10^{-2}$	$1.4\ 10^{-2}$	$5.6\ 10^{-3}$	$9.3\ 10^{-3}$
17	$1.7\ 10^{-2}$	$1.4\ 10^{-2}$	$6.4\ 10^{-3}$	$9.5\ 10^{-3}$

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