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The Lower Bounds of Eigenvalues by the Wilson Element in Any Dimension

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Abstract. In this paper, we analyze the Wilson element method of the eigenvalue problem in arbitrary dimensions by combining a new technique recently developed in [10] and the a posteriori error result. We prove that the discrete eigenvalues are smaller than the exact ones.

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Key words: The lower approximation, the Wilson element, the eigenvalue problem.

1 The Wilson element in any dimension

This paper is devoted to the finite element approximation of the following elliptic eigenvalue problem: find $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ with

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \lambda(\rho u, v)_{L^2(\Omega)}, \text{ for any } v \in H_0^1(\Omega), \text{ with } \big\|\rho^{\frac{1}{2}} u\big\|_{L^2(\Omega)} = 1, \tag{1.1}$$

where $\rho \in L^\infty(\Omega)$ is a positive function.

Let $\mathcal{T}_h$ be a regular n-rectangular triangulation of the n-rectangular domain $\Omega \subset \mathbb{R}^n$ with $2 \leq n$ in the sense that

$$\bigcup_{K \in \mathcal{T}_h} K = \bar{\Omega},$$

two distinct elements K and K' in $\mathcal{T}_h$ are either disjoint, or share the ℓ-dimensional hyper-plane, $\ell = 0, \cdots, n-1$. Let $\mathcal{H}_h$ denote the set of all $n-1$ dimensional hyper-planes in $\mathcal{T}_h$ with the set of interior $n-1$ dimensional hyper-planes $\mathcal{H}_h(\Omega)$ and the set of boundary $n-1$ dimensional hyper-planes $\mathcal{H}_h(\partial\Omega)$. We let $\mathcal{N}_h$ denote the set of nodes of $\mathcal{T}_h$ with the set of internal nodes $\mathcal{N}_h(\Omega)$ and the set of boundary nodes $\mathcal{N}_h(\partial\Omega)$.

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For each $K \in \mathcal{T}_h$, we introduce the following affine invertible transformation

$$F_K:\ \hat{K} \to K,\ \ x_i = h_{x_i,K}\xi_i + x_i^0,$$

with the center $(x_1^0, x_2^0, \cdots, x_n^0)$ and the lengths $2h_{x_i,K}$ of K in the directions of the x_i-axis, and the reference element $\hat{K} = [-1,1]^n$. In this paper, we only consider the uniform mesh with $h_{x_i} = h_{x_i,K}$ for any $K \in \mathcal{T}_h$. In addition, we set $h = \max_{1\le i\le n} h_{x_i}$.

Denote by $Q_{nD}(\hat{K})$ the nonconforming Wilson element space [17] on the reference element defined by

$$Q_{nD}(\hat{K}) = Q_1(\hat{K}) + \text{span}\left\{\xi_1^2 - 1, \xi_2^2 - 1, \cdots, \xi_n^2 - 1\right\}, \tag{1.2}$$

where $Q_1(\hat{K})$ is the space of polynomials of degree≤ 1 in each variable. The nonconforming Wilson element space V_h^{nc} is then defined as

$$V_h^{nc} := \Big\{ v \in L^2(\Omega):\ v|_K \circ F_K \in Q_{nD}(\hat{K}) \text{ for each } K \in \mathcal{T}_h,\ v \text{ is continuous at the internal nodes, and vanishes at the boundary nodes} \Big\}.$$

Define the discrete semi-norm on V_h^{nc} by

$$|v|_h^2 = \sum_{K\in\mathcal{T}_h} \|\nabla v\|^2_{L^2(K)}.$$

By the Poincare inequality, we have $|\cdot|_h$ as a norm on V_h^{nc}. The finite element approximation of Problem (1.1) reads: find $(\lambda_h, u_h) \in \mathbb{R} \times V_h^{nc}$, such that

$$(\nabla_h u_h, \nabla_h v_h)_{L^2(\Omega)} = \lambda_h(\rho u_h, v_h)_{L^2(\Omega)}, \text{ for any } v_h \in V_h^{nc}, \text{ with } \big\|\rho^{\frac{1}{2}} u_h\big\|_{L^2(\Omega)} = 1. \tag{1.3}$$

The purpose of this paper is to analyze the lower approximation property of eigenvalues produced in (1.3). By combining the method based on the identity from [1,11] and the technique developed for the Adini element in a recent paper [10], we prove that the discrete eigenvalues are smaller than the exact ones when the meshsize h is small enough. Compared to the result of [19] only for the three dimensions, the novelties of the paper are of twofold: It analyzes the Wilson element in any dimension [17]; it is able to weaken the regularity condition on the eigenfunction.

The rest of the paper is organized as follows. In the following section, we prove the main result of this paper, ie., the discrete eigenvalues produced by the Wilson element are smaller than the exact ones. Some proof details are presented in Section 3.

2 Lower approximations of eigenvalues

We show that the approximate eigenvalues are smaller than the exact ones in this section. We first define the canonical interpolation. Let

$$a_i = (\xi_{1i}, \xi_{2i}, \cdots, \xi_{ni}), \qquad i = 1, \cdots, 2^n,$$