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A Boundary Meshless Method for Solving Heat Transfer Problems Using the Fourier Transform

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Abstract. Fourier transform is applied to remove the time-dependent variable in the diffusion equation. Under non-harmonic initial conditions this gives rise to a non-homogeneous Helmholtz equation, which is solved by the method of fundamental solutions and the method of particular solutions. The particular solution of Helmholtz equation is available as shown in [4, 15]. The approximate solution in frequency domain is then inverted numerically using the inverse Fourier transform algorithm. Complex frequencies are used in order to avoid aliasing phenomena and to allow the computation of the static response. Two numerical examples are given to illustrate the effectiveness of the proposed approach for solving 2-D diffusion equations.

AMS subject classifications: 65Y04, 35K05

Key words: Transient heat transfer, meshless methods, method of particular solutions, method of fundamental solutions, frequency domain, Fourier transform.

1 Introduction

Over the past four decades researchers have proposed a variety of numerical techniques to solve heat transfer problems. The Finite Element Method (FEM) and the Finite Difference Method (FDM) are well-established techniques that have often been implemented to solve these types of problems. However, they require the discretization of the full domain, which leads to problems that can be tedious and computationally costly, particularly for unbounded or high dimensional irregular domains. Different numerical techniques, such as the Boundary Element Method (BEM), have been

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developed to alleviate these computational difficulties by reducing the discretization of the problem domain to the material interfaces. However, the BEM requires the prior knowledge of fundamental solutions, and it leads to integrations along boundary elements that may be singular or even hyper-singular. Furthermore, the discretization of a three-dimensional surface is still not an easy task. More recently, attention has been focused on the development of meshless methods which require neither domain nor boundary discretization. Among these methods, the method of fundamental solutions (MFS) has emerged as an effective boundary-only meshless method for solving homogeneous equations [7,9,11]. Coupled with radial basis functions (RBFs), the MFS can be extended to solve nonhomogeneous equations, nonlinear equations, and time-dependent problems [5–7, 10–12].

Most of the techniques that have been implemented to solve transient heat transfer problems use time marching schemes [1, 3, 14, 17, 20], or Laplace transforms [5, 18, 19, 21]. The Laplace transform technique replaces the time dependence by a transform variable. However, the numerical inverse Laplace transform is ill-posed, which means that small truncation errors are magnified in the numerical inversion process. Despite the progress in numerical inversion techniques of the Laplace transfrom in recent years, the difficulty remains. The purpose of this paper is to apply the Fourier transform to remove the time dependent variable. As a result, the given heat transfer problem is reduced to a nonhomogeneous Helmholtz equation, which can be solved using the boundary meshless method mentioned above. The method of particular solution is implemented to solve nonhomogeneous Helmholtz equation in the frequency domain. In this process, the derivation of close form particular solution is crucial and is not an easy task. A particular solution for Helmholtz-type equations was originally proposed by Chen and Rashed [4] using thin plate splines and later generalized to polyharmonic splines by Muleshkov et al. [15]. The homogeneous solution is obtained by the standard MFS. Finally, time solutions are obtained by applying an inverse Fourier transform algorithm. To avoid aliasing phenomena, complex frequencies are introduced into the problem which also allows the computation of the static response. The effect of the presence of these complex frequencies is removed in the time domain by using an exponential window to rescale the response.

In Section 2 we first define the two-dimensional heat transfer problems and describe how to convert it to Helmholtz equation using Fourier transform. In Section 3, we briefly explain how to obtain the particular solution using thin plate splines. In Section 4, the MFS for solving the homogeneous solutions is described. In Section 5, we give a brief account of how the time solutions are obtained using inverse Fourier transform. In Section 6, the proposed technique is verified by solving two different heat transfer problems in a rectangular domain for which analytical solutions are known. We first consider constant initial conditions in the full domain and the verification procedure is performed in the time domain, while the second problem assumes a non-constant temperature distribution in the full inner domain. For this latter case, other than time domain comparisons against explicit results, verifications of the responses in the frequency domain are also presented.

2 Fourier transform to remove the time variable

Let Ω be a bounded domain in $\mathbf{R}^2$ with boundary

$$S = S_1 \cup S_2, \quad S_1 \cap S_2 = \emptyset.$$

The following transient heat transfer by conduction in a homogeneous and isotropic body can be modeled by

$$\frac{1}{K}\frac{\partial T}{\partial t} = \nabla^2 T(x,y,t) + g(x,y,t), \quad (x,y) \in \Omega, \quad t > 0, \tag{2.1}$$

with boundary conditions

$$T(x,y,t) = f_1(x,y,t), \qquad (x,y) \in S_1, \quad t > 0, \tag{2.2a}$$

$$\frac{\partial T(x,y,t)}{\partial n} = f_2(x,y,t), \qquad (x,y) \in S_2, \quad t > 0, \tag{2.2b}$$

and initial condition

$$T(x,y,0) = T_0(x,y), \quad (x,y) \in \Omega, \tag{2.3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

t is the time, $T(x,y,t)$ is the temperature, $T_0(x,y)$ is initial condition, $K{=}k/\rho c$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density, c is the specific heat, and $f_1(x,y,t)$ and $f_2(x,y,t)$ are given boundary conditions.

To solve this problem we convert the problem from the time domain to the frequency domain, by applying the Fourier transform with respect to t in (2.1). Using the Fourier transform with respect to t, $\mathcal{F}_t[T(x,y,t)]$, we have

$$\hat{T}(x,y,\omega) = \mathcal{F}_t\big[T(x,y,t)\big] = \int_0^\infty T(x,y,t)e^{\mathrm{i}\omega t}dt, \tag{2.4}$$

where ω is the Fourier spectral parameter or frequency. By standard procedure of integration by parts, we obtain

$$(\nabla^2 + \lambda^2)\hat{T}(x,y,\omega) = -\frac{T_0(x,y)}{K} + \hat{g}(x,y,\omega), \qquad (x,y) \in \Omega, \tag{2.5a}$$

$$\hat{T}(x,y,\omega) = \hat{T}_D(x,y,\omega), \qquad (x,y) \in S_1, \tag{2.5b}$$

$$\frac{\partial \hat{T}(x,y,\omega)}{\partial n} = \hat{T}_N(x,y,\omega), \qquad (x,y) \in S_2, \tag{2.5c}$$

where

$$\lambda = \sqrt{-\frac{\mathrm{i}\omega}{K}}, \qquad \hat{g}(x,y,\omega) = \mathcal{F}_t[g(x,y,t)],$$

$$\hat{T}_D(x,y,\omega) = \mathcal{F}_t[f_1(x,y,t)], \qquad \hat{T}_N(x,y,\omega) = \mathcal{F}_t[f_2(x,y,t)].$$

Eqs. (2.5a)-(2.5c) can be solved using the method of particular solutions (MPS), which is a well-known technique for solving ordinary and partial differential equations. Let us define

$$\hat{T}(x,y,\omega) = \hat{T}_p(x,y,\omega) + \hat{T}_h(x,y,\omega), \tag{2.6}$$

where $\hat{T}_p(x,y,\omega)$ is a particular solution that satisfies the equation

$$(\nabla^2 + \lambda^2)\hat{T}_p(x,y,\omega) = -\frac{T_0(x,y)}{K} + \hat{g}(x,y,\omega), \tag{2.7}$$

but does not necessarily satisfy the boundary conditions. Therefore, $\hat{T}_h(x,y,\omega)$ satisfies the homogeneous equation,

$$(\nabla^2 + \lambda^2)\hat{T}_h(x,y,\omega) = 0, \qquad (x,y) \in \Omega, \tag{2.8a}$$

$$\hat{T}_h(x,y,\omega) = \hat{T}_D(x,y,\omega) - \hat{T}_p(x,y,\omega), \qquad (x,y) \in S_1, \tag{2.8b}$$

$$\frac{\partial \hat{T}_h(x,y,\omega)}{\partial n} = \hat{T}_N(x,y,\omega) - \frac{\partial \hat{T}_p(x,y,\omega)}{\partial n}, \qquad (x,y) \in S_2. \tag{2.8c}$$

Thus, once the particular solution is known, the homogeneous equation (2.8a)-(2.8c) can be solved using various types of boundary methods such as the boundary integral equation method (BIE), the boundary element method (BEM) [20], or the method of fundamental solutions (MFS) [7, 9, 11]. To obtain a mesh free method, the MFS is employed for solving (2.8a)-(2.8c). We would like to note that the evaluation of particular solution is critical to the accuracy of the overall solution. The approximation error introduced in (2.8b)-(2.8c) could be magnified during the computational process for solving homogeneous solution.

Once $\hat{T}(x,y,\omega)$ in frequency domain has been determined, the final solution $T(x,y,t)$ in the time domain can be obtained using inverse Fourier transform which is defined as follows

$$T(x,y,t) = \mathcal{F}_t^{-1}\big[\hat{T}(x,y,\omega)\big] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{T}(x,y,\omega)e^{-\mathrm{i}\omega t}d\omega.$$

The solution of this integral is obtained after discretization. The resulting inverse discrete Fourier transformation corresponds to the addition of sources at time intervals of $T{=}2\pi/\Delta\omega$ (where $\Delta\omega$ is the frequency increment).

The contamination of the response by the periodic virtual sources (i.e., aliasing phenomena) is avoided by setting frequency increments small enough to guarantee that the dynamic contribution of the response arrives within the time interval T. This is further helped by introducing complex frequencies with a small imaginary part of the form $\omega_c{=}\omega - \mathrm{i}\eta$ (with $\eta{=}0.7\Delta\omega$), which shifts the frequency axis slightly downwards in the complex plane. When the time responses are finally evaluated, the effect of using complex frequencies must be taken into account by rescaling the responses with an exponential factor $e^{\eta t}$.

In essence, the solution involves: (1) finding the response within the frequency domain defined by the forward Fourier transform of the excitation source, with a frequency increment that defines the time window, for complex frequencies in the range from 0.0Hz up to frequencies where the response is negligible; in heat diffusion phenomena the contribution to the response decreases rapidly as the frequency increases; (2) performing a standard inverse discrete Fourier transformation into the time domain; and (3) removing the effect of the complex frequencies by means of an exponential factor (or window). This computational advantage is achieved at the expense of having to evaluate accurately the responses at each frequency step, since interpolation schemes cannot be used in this method.

3 The particular solution using thin plate splines

Before obtaining the homogeneous solution in (2.8a)-(2.8c), we need to obtain a particular solution in (2.7). The derivation of particular solution for Helmholtz equation plays a critical role in the process of boundary-only meshless methods. Traditionally, due to the difficulty of obtaining particular solutions, the Laplace operator has been largely used as the main differential operator while other terms of the governed differential operator were moved to the right hand side of the equation and treated as part of the forcing term [16]. For instance, the Helmholtz equation in (2.7) is reformulated as

$$\nabla^2 \hat{T}_p(x,y,\omega) = -\lambda^2 \hat{T}_p(x,y,\omega) - \frac{T_0(x,y)}{K} + \hat{g}(x,y,\omega). \tag{3.1}$$

The dual reciprocity boundary elememnt method (DRBEM) [16] is suitable for such a purpose. When using the Laplace operator in (3.1) instead of the Helmholtz operator in (2.7) as a whole for the evaluation of particular solutions, some information in the governing equation may be lost and the forcing terms $-\lambda^2 \hat{T}_p(x,y,\omega) - T_0(x,y)/K + \hat{g}(x,y,\omega)$ in (3.1) become more difficult to approximate by radial basis functions. It is desirable to approximate the particular solution directly from (2.7) using the Helmholtz operator. Due to the fact that we can find the closed-form approximate particular solution for Helmholtz-type equations [4,15], significant improvement for solving various types of differential equations has been reported [5,7,11,12].

We give a brief review on how to obtain a closed-form particular solution for Helmholtz equation (2.7). First, the nonhomogeneous term $-T_0(x,y)/K + \hat{g}(x,y,\omega)$ in (2.7) is approximated by a linear combination of radial basis functions, in particular thin plate splines; i.e.,

$$-\frac{T_0(x,y)}{K} + \hat{g}(x,y,\omega) \simeq F(x,y) = \sum_{i=1}^{N} \alpha_i r_i^2 \log r_i + a + bx + cy, \tag{3.2}$$

with the further constraints

$$\sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} \alpha_i x = \sum_{i=1}^{N} \alpha_i y = 0, \tag{3.3}$$

where

$$r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2},$$

and $\{(x_i, y_i)\}_{i=1}^N$ are the interpolation points scattered in the domain. The polynomial terms of (3.2) and constraints in (3.3) are added to ensure that

$$\begin{cases} -\dfrac{T_0(x_j,y_j)}{K} + \hat{g}(x_j,y_j,\omega) = F(x_j,y_j), & 1 \le j \le N, \\ \sum\limits_{i=1}^N \alpha_i = \sum\limits_{i=1}^N \alpha_i x_j = \sum\limits_{i=1}^N \alpha_i y_j = 0, & 1 \le j \le N, \end{cases} \tag{3.4}$$

is uniquely solvable [8]. Radial basis functions are very effective tool in high dimensional surface interpolation. By the collocation method, a system of $N+3$ linear equations can be formulated and solved. Once the α_i, a, b and c are obtained, we have

$$(\nabla^2 + \lambda^2)\hat{\mathcal{T}}_p(x,y,\omega) = \sum_{i=1}^N \alpha_i r_i^2 \log r_i + a + bx + cy, \tag{3.5}$$

where $\hat{\mathcal{T}}_p(x,y,\omega)$ is an "approximate" particular solution. The closed form $\hat{\mathcal{T}}_p(x,y,\omega)$ is available as follows [4,15]

$$\hat{\mathcal{T}}_p(x,y,\omega) = \sum_{i=1}^N \alpha_i \Phi(r) + \frac{a}{\lambda^2} + \frac{bx}{\lambda^2} + \frac{cy}{\lambda^2}, \tag{3.6}$$

where

$$\Phi(r) = \begin{cases} \dfrac{2}{\lambda^4}\left[\pi Y_0(\lambda r) - 2\log r\right] + \dfrac{r^2 \log r}{\lambda^2} - \dfrac{4}{\lambda^4}, & r > 0, \\ -\dfrac{2}{\lambda^4}\left[\gamma + \log\left(\dfrac{\lambda}{2}\right)\right] - \dfrac{4}{\lambda^4}, & r = 0, \end{cases} \tag{3.7}$$

where $\gamma \simeq 0.5772156649015328$ is the Euler's constant and Y_0 is the Bessel function of the second kind with order zero.

4 The MFS for Helmholtz equation

Using the MFS we approximate the homogeneous solution by placing M source points $\{(t_j, s_j)\}_{j=1}^M$ on a fictitious boundary outside the domain to avoid singularities of the fundamental solutions. For details, we refer reader to some recent review papers in the MFS [7,9,11]. The approximate homogeneous solution of (2.8a)-(2.8c), $\hat{\mathcal{T}}_h$ to $\hat{T}_h$ can be represented as a linear combination of fundamental solutions

$$\hat{\mathcal{T}}_h(x,y,\omega) = \sum_{j=1}^M a_j G(x,y,t_j,s_j,\omega), \tag{4.1}$$

where

$$G(x,y,t_j,s_j,\omega)=\frac{-\mathrm{i}}{4k}H_0\Big(\sqrt{\frac{-\mathrm{i}\omega}{K}}r\Big), \tag{4.2}$$

in which

$$r=\sqrt{(x-t_j)^2+(y-s_j)^2},$$

and H_0 are Hankel functions of the second kind with order zero. Since (4.1) satisfies the governing equation, we need only to fit the boundary conditions in (2.8b) and (2.8c). As we shall see, the solution procedure is straightforward.

Once the source points have been chosen, the coefficients $\{a_j\}_{j=1}^M$ can be obtained by satisfying the boundary conditions along the physical boundary S. In the collocation method, we choose the number of source points on the fictitious boundary and the number of collocation points on the boundary S to be equal. Let $\{(x_i,y_i)\}_{i=1}^{M_1}$ be the collocation points on S_1 and $\{(x_i,y_i)\}_{i=M_1+1}^{M}$ the points on S_2. In general, the collocation points $\{(x_j,y_j)\}_{j=1}^M$ are uniformly distributed on the physical boundary. By substituting (4.1) into (2.8b) and (2.8c), we have

$$\hat{T}_D(x_i,y_i,\omega)-\hat{T}_p(x_i,y_i,\omega)=\sum_{j=1}^{M}a_jG(x_i,y_i,t_j,s_j,\omega), \qquad 1\le i\le M_1, \tag{4.3a}$$

$$\hat{T}_N(x_i,y_i,\omega)-\frac{\partial\hat{T}_p(x_i,y_i,\omega)}{\partial n}=\sum_{j=1}^{M}a_j\frac{\partial G(x_i,y_i,t_j,s_j,\omega)}{\partial n}, \quad M_1+1\le i\le M. \tag{4.3b}$$

The final system of M Eqs. (4.3a)-(4.3b) can be solved by Gaussian elimination or other linear solvers.

Once the $\{a_j\}_{j=1}^M$ are found, $\hat{\mathcal{T}}_h$ can be evaluated from (4.1) at any point in the domain. Thus, an approximation solution $\hat{\mathcal{T}}(x,y,\omega)$ to $\hat{T}(x,y,\omega)$ in the frequency domain can be obtained as follows

$$\hat{\mathcal{T}}(x,y,\omega)=\hat{\mathcal{T}}_h(x,y,\omega)+\hat{\mathcal{T}}_p(x,y,\omega). \tag{4.4}$$

5 Responses in the time domain

The heat responses in the time domain are computed by applying inverse Fourier transforms in the frequency domain. Aliasing phenomena are prevented by the use of complex frequencies of the form $\omega_c=\omega-\mathrm{i}\eta$ (with $\eta=0.7\Delta\omega$, and $\Delta\omega$ being the frequency step). The constant η cannot be made arbitrarily large, since this leads to serious failure of numerical precision (see Kausel and Roësset [13]). The required static response can be computed when the frequency is zero, since the use of complex frequencies leads to arguments of the Hankel function other than zero ($\omega_c=-\mathrm{i}\eta$ for 0.0Hz).

6 Verification of the solution

To verify the accuracy of the formulation described above, we consider the diffusion equation in (2.1) without a source term, i.e.,

$$g(x,y,t)=0,$$

in a finite rectangular domain

$$\Omega=\big\{(x,y): -a\le x\le a,\ -b\le y\le b\big\},$$

in which nonzero initial temperatures are prescribed inside the domain, maintaining the boundaries at zero temperature. Here, we choose a=b=0.2m.

The thermal properties of the homogeneous medium are k=63.9 W/mC, ρ=7832.0 kg/m^3 and c=434.0 J/kgC, which defines a thermal diffusivity of K=1.88 $\times$ 10^{-5}m^{2}s^{-1}.

Two initial temperature distributions were considered, for which exact solutions are known. We first consider a unit initial temperature distribution $T_0(x,y)$=1.0°C for $(x,y)\in\Omega\backslash\partial\Omega$. Note that there is a jump in the temperature distribution in the initial stage. The analytical temperature distribution is given by Carslaw and Jaeger [2]:

$$T(x,y,t)=\frac{16}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}L_{n,m}\cos\frac{(2n+1)\pi x}{2a}\cos\frac{(2m+1)\pi y}{2b}\exp(-D_{n,m}t), \tag{6.1}$$

where

$$L_{n,m}=\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\quad\text{and}\quad D_{n,m}=\frac{k\pi^2}{4}\Big[\frac{(2n+1)^2}{a^2}+\frac{(2m+1)^2}{b^2}\Big].$$

In the second example, we consider the initial temperature distribution as follows

$$T_0(x,y)=\cos\Big(\frac{\pi}{2a}x\Big)\cos\Big(\frac{\pi}{2b}y\Big),\quad (x,y)\in\Omega. \tag{6.2}$$

The exact frequency responses are given by

$$T(x,y,\omega)=T_0(x,y)\Big\{K\Big[\Big(\frac{\pi}{2}\Big)^2\Big(\frac{1}{a^2}+\frac{1}{b^2}\Big)\Big]+\mathrm{i}\omega\Big\}^{-1}, \tag{6.3}$$

and the time domain solution is

$$T(x,y,t)=T_0(x,y)\exp\Big[-K\Big(\frac{\pi}{2}\Big)^2\Big(\frac{1}{a^2}+\frac{1}{b^2}\Big)t\Big]. \tag{6.4}$$

Using the proposed formulation, the solutions in time domain were obtained after the computation of the responses in the frequency domain in the range $[0,2048\times10^{-3}]$Hz with an increment of 25×10^{-5}Hz, which defines a time window of T=4000s.

6.1 The case $T_0(x,y)$ =1.0°C

The heat responses were calculated on a line of receivers crossing the rectangular domain at y=0.0m and at a receiver placed at $(0.15, 0.15)$m. Fig. 1 illustrates the positions of these receivers.

As $T_0(x,y)$ is constant, the particular solution $\hat{T}_p(x,y,\omega)$ was obtained using only one internal point, as shown in Fig. 1. This is due to the fact that constant function can be fitted precisely with one point. As a result, $T_0(x,y)$ can be approximated perfectly using the augmented polynomial terms of thin plate splines in (3.2) which can reproduce the constant and linear functions. In this case, no approximation error is expected; i.e.,

$$\hat{T}_p(x,y,\omega) = T(x,y,\omega).$$

The homogeneous solution $\hat{T}_h(x,y,\omega)$ is obtained using the MFS with 400 source points uniformly distributed on a square, at a distance of 0.05m from the boundary, as shown in the layout in Fig. 1. The same number of collocation points that are uniformly distributed on the boundary is used.

Fig. 2 shows the temperature values obtained at different times ($t = 0, 125, 250, 500, 750, 1000, 1500$s), for the line of receivers equally spaced at 0.01m, placed at y=0.0m (see Fig. 1). The solid lines represent the solution given by the Eq. (6.1) while the circles identify the response provided by the proposed formulation. These results are very similar and in good agreement. As time evolves the temperatures drop very fast in the domain in order to establish equilibrium with the boundary conditions (null temperatures).

Fig. 3 presents the temperature curves on five different receivers. It displays the exact and numerical solutions in the time domain. The numerical responses are represented by different symbols. Each symbol identifies a specific receiver (coordinates

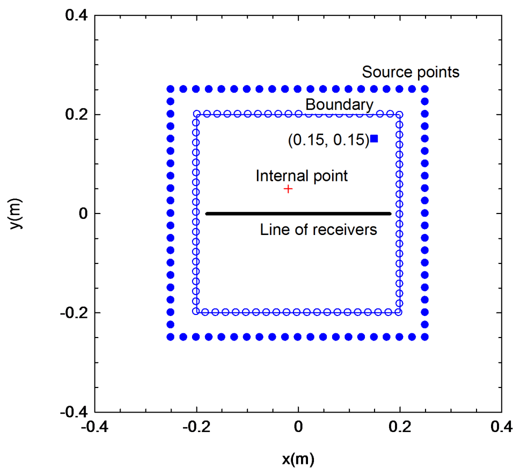

Figure 1: Geometry of the problem. Source, internal points and positions of receivers for the case $T_0(x,y)$ =1.0°C.

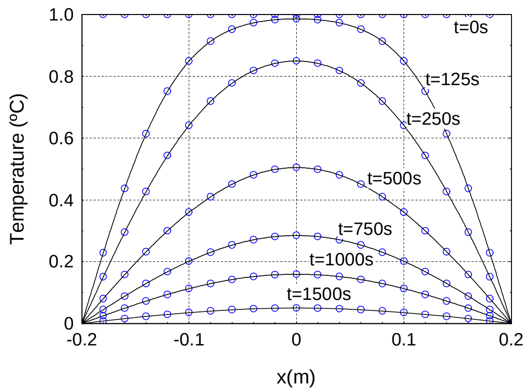

Figure 2: Exact and numerical solutions for $T_0(x,y)$ =1.0°C. Heat responses for a horizontal line of receivers (placed at $y=0$m) at $t=0,125,500,750,1000,1250$ and 1500s.

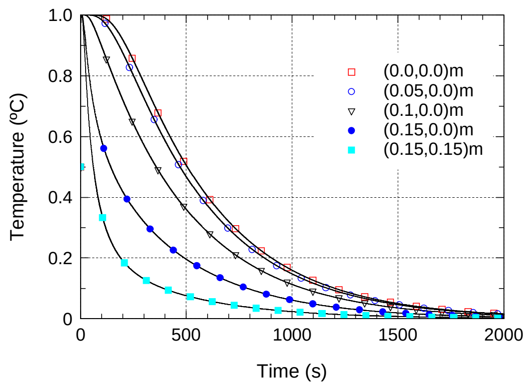

Figure 3: Analytical versus numerical transient solutions on five receivers for $T_0(x,y)$ =1.0°C.

are given in the legend).

Notice that, at t=0.0s, the numerical solutions do not coincide with the exact solutions due to the fact that the $T(x,y,t)$ obtained from a Fourier transform is discontinuous at this instant.

6.2 The case $T_0(x,y)=\cos(\frac{\pi}{2a}x)\cos(\frac{\pi}{2b}y)$

In this case, the distribution of the initial temperature is more complicated and thus more internal points are required to interpolate the initial condition using thin plate splines interpolation scheme. We choose 120 internal points to perfom such an interpolation (see Fig. 4). It has been observed that the convergence of the interpolation function used in the definition of the particular solution depends on the distribution of the intepolation points, in particular at higher frequencies. As the frequency increases the distance between the internal points and the boundary needs to be increased, as illustrated in Fig. 4 for two different frequencies ω=0.0Hz (left) and 0.0125Hz (right).

The homogeneous solution was calculated using 400 source points evenly distributed at a distance of 0.05m around the boundary. The same number of collocation

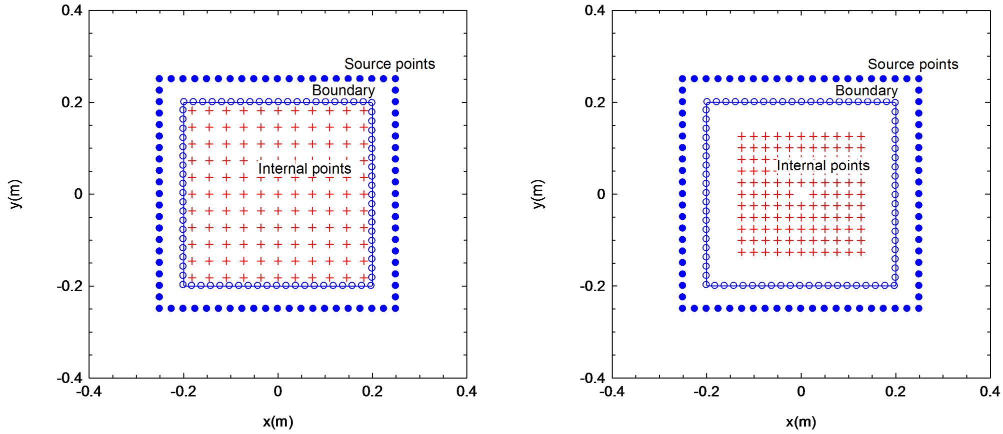

Figure 4: Geometry of the problem. Source and internal points at $\omega = 0.0$Hz (left) and at $\omega = 1.25 \times 10^{-2}$Hz (right) for the case $T_0(x,y) = \cos(\frac{\pi}{2a}x)\cos(\frac{\pi}{2b}y)$.

points are also evenly distributed on the physical boundary.

The numerical results in the frequency domain, which is obtained using the proposed formulation, were compared with the explicit solutions given in (6.3). We select five receivers at the position $R1{=}(0,0)$m, $R2{=}(0.05,0)$m, $R3{=}(0.1,0)$m, $R4{=}(0.15,0)$m and $R5{=}(0.15,0.15)$m. Analytical and numerical solutions in the frequency domain at these five receivers are shown in Figs. 5-7 in the frequency range $[0, 5.0 \times 10^{-3}]$Hz. This verification shows good agreement between the numerical values (marks) and the explicit responses (solid lines).

The application of an inverse Fourier transform in the frequency domain allows the solution in the time domain to be obtained. Fig. 8 presents the results of temperature curves at these five receivers. The calculations were performed over the frequency range $[0, 2048 \times 10^{-3}]$Hz, prescribing a frequency increment of 25×10^{-5}Hz. Fig. 8 also shows the temperatures calculated using Eq. (6.4), which are represented by marks.

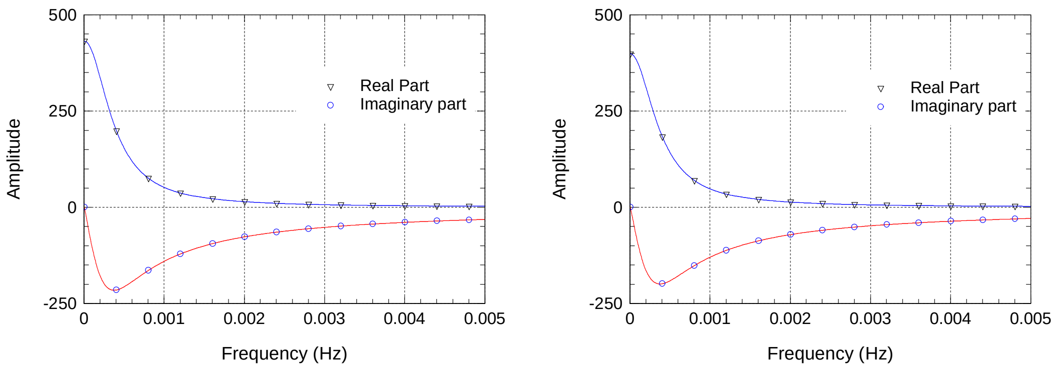

Figure 5: Analytical and numerical solutions in the frequency domain at $R1 = (0,0)$m (left) and $R2 = (0.05,0)$m (right).

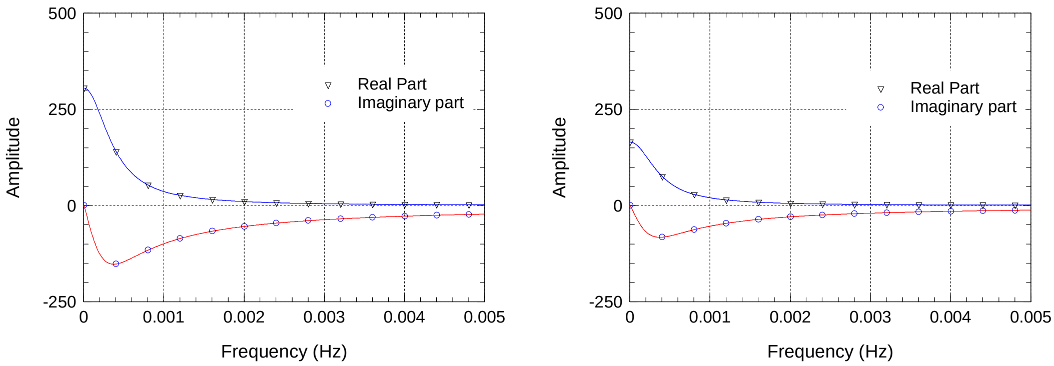

Figure 6: Analytical and numerical solutions in the frequency domain at $R3 = (0.1, 0)$m (left) and $R4 = (0.15, 0)$m (right).

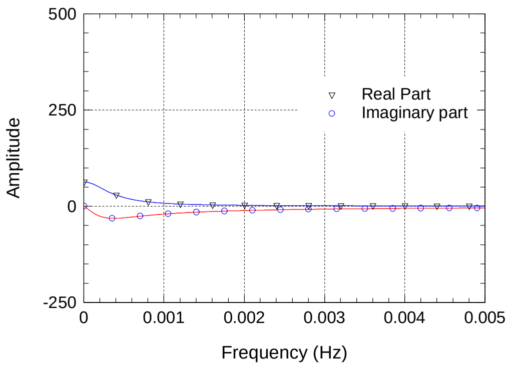

Figure 7: Analytical and numerical time solutions in the frequency domain at $R5 = (0.15, 0.15)$m.

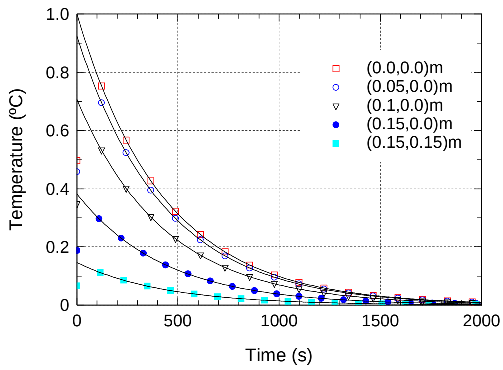

Figure 8: Analytical and numerical transient solutions at five receivers for the case $T_0(x, y) = \cos(\frac{\pi}{2a}x)\cos(\frac{\pi}{2b}y)$.

The numerical and analytical results in time domain are also in good agreement.

Once again, at t=0.0s, the numerical solutions are positioned at half the amplitude of the exact time values due to the fact that the Fourier transform of $T(x, y, t)$ is discontinuous at this instant.

7 Conclusions

In this paper we have demonstrated how the solution of diffusion equations with nonzero initial conditions can be computed using Fourier transform which temporarily removes the time-dependent variable. This leads to a nonhomogeneous Helmholtz equation. The MFS coupled with the method of particular solution were employed to solve the nonhomogeneous Helmholtz equation. A particular solution for Helmholtz equation using thin plate splines is available, see [4, 15]. The main distinction of this paper with previous works [5, 21] is the employment of Fourier transform instead of Laplace transform to remove the time-dependent variable. As a result, instead of solving modified Helmholtz equation, we have to deal with Helmholtz equation which involves a complex fundamental solution.

We would like to remark that a similar solution process can be extended to solving a large class of time-dependent problems such as wave equations, convection-diffusion equations, and various types of nonlinear differential equations.

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