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Dependence Analysis of the Solutions on the Parameters of Fractional Delay Differential Equations

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Abstract. In this paper, we investigate the dependence of the solutions on the parameters (order, initial function, right-hand function) of fractional delay differential equations (FDDEs) with the Caputo fractional derivative. Some results including an estimate of the solutions of FDDEs are given respectively. Theoretical results are verified by some numerical examples.

AMS subject classifications: 34A08, 34A10, 34A12, 34K37
Key words: Fractional delay differential equation, Caputo fractional derivative, dependence.

1 Introduction

In the recent past years, the use of differential equations of fractional order (FDEs) has gained considerable popularity in several areas such as nonlinear oscillation of earthquake (cf. [1]), fluid-dynamic traffic model (cf. [2]), material viscoelastic theory and physics (cf. [3–6]), etc. In [8,9], some results about the dependence of the solutions on the parameters (including the order of the differential equation, the initial function and the right-hand function) of some classes of fractional differential equations (FDEs) with Riemann-Liouville (R-L) fractional derivatives were given.

In this paper, our aim is to extend some results about the dependence in [9] and to consider the dependence of the solutions on the above parameters of the following initial value problem of a FDDE in the form

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = f(t, x_t), & t \in [0, T], \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

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where $0 < \alpha < 1$, $\tau > 0$, $f : D = [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t,x_t) = f(t,x(t),x(t-\tau))$, and ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order α and is defined in [6] as

$$ {}_0^C D_t^\alpha y(t) = I^{1-\alpha}\frac{dy(t)}{dt} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{dy(\tau)}{d\tau}d\tau, \quad t>0, \tag{1.2} $$

where I^α denotes the integral operator of order α and is defined in [6] as

$$ I^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau, \quad t>0, \quad \alpha>0. $$

As we all know, there are some different definitions of fractional operator except the Caputo fractional derivative. From a theoretical point of view the most natural approach is the Riemann-Liouville definition defined in [6] as

$$ {}_0^R D_t^\alpha y(t) = \frac{d}{dt}(I^{1-\alpha}y(t)) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}y(\tau)d\tau, \quad t>0. \tag{1.3} $$

The relationship between the Caputo definition and the Riemann-Liouville definition can be given by the following formula (cf. [5])

$$ {}_0^R D_t^\alpha y(t) = {}_0^C D_t^\alpha y(t) + \frac{y(0)}{\Gamma(1-\alpha)}t^{-\alpha}, \quad t>0. \tag{1.4} $$

Thus, the Problem (1.1) can be written as

$$ \begin{cases} {}_0^R D_t^\alpha (x(t)-x(0)) = f(t,x_t), & t\in[0,T], \\ x(t) = \varphi(t), & t\in[-\tau,0]. \end{cases} \tag{1.5} $$

As in [7], the Problem (1.5) have the following form

$$ \begin{cases} x(t) = \varphi(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(s,x_s)ds, & t\in[0,T], \\ x(t) = \varphi(t), & t\in[-\tau,0]. \end{cases} \tag{1.6} $$

In [7], existence and uniqueness of solutions to Problems (1.1), (1.5), (1.6) were given. In this paper, we assume that the existence and uniqueness of solution of Problem (1.1) hold.

This paper is organized as follows. In Section 2, some results about dependence of the solutions on the parameters of FDDEs are given, and we also give the estimate of the solutions of FDDEs. In Section 3, we identify our some theoretical results by some examples.

2 The main results

In this section, we shall present and prove our main results. Firstly, we introduce the following Lemmas and define the norm

$$ \|u(t)\|_\infty = \max_{0\le t\le T}|u(t)|, \quad \forall u(t)\in C[0,T]. $$

Lemma 2.1. *(see* [8]*) Suppose that $\alpha > 0$, $b \geq 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\Big]ds, \quad 0 \leq t < T.$$

Furthermore, if $a(t)$ is nondecreasing function on $[0,T)$, then

$$u(t) \leq a(t)E_\alpha(b\Gamma(\alpha)t^\alpha),$$

where E_α is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}.$$

Lemma 2.2. *(see* [9]*) If $k(x)$ is a continuous function, and $\epsilon > 0$, $0 < \alpha - \epsilon < \alpha < 1$, then*

$${}_0^RD_t^\epsilon k(x) = k(x) + \mathcal{O}(\epsilon)k(x).$$

When the order α is perturbed by a small parameter ϵ, we can obtain the following theorem for the corresponding solution perturbation.

Theorem 2.1. *Assume that Problem* (1.5) *satisfies the Lipschitz condition*

$$\begin{aligned}&|f(t,x(t),y(t)) - f(t,\tilde{x}(t),\tilde{y}(t))|\\ \leq&\gamma|x(t)-\tilde{x}(t)| + \beta|y(t)-\tilde{y}(t)|, \quad \forall x,\ \tilde{x},\ y,\ \tilde{y} \in C[0,T],\ \ t\in[0,T],\end{aligned} \tag{2.1}$$

where γ, β are Lipschitz constants. If $y(t)$ and $z(t)$ are the uniquely determined solutions of the following problems

$$\begin{cases} {}_0^RD_t^{\alpha-\epsilon}(y(t)-y(0)) = f(t,y_t), & t\in[0,T],\\ y(t) = \varphi(t), & t\in[-\tau,0],\end{cases} \tag{2.2}$$

and

$$\begin{cases} {}_0^RD_t^{\alpha}(z(t)-z(0)) = f(t,z_t), & t\in[0,T],\\ z(t) = \varphi(t), & t\in[-\tau,0],\end{cases} \tag{2.3}$$

respectively, where $\epsilon > 0$, $0 < \alpha - \epsilon < \alpha < 1$, then we have the relation

$$\|y(t) - z(t)\|_\infty = \mathcal{O}(\epsilon).$$

Proof. Obviously, according to Problems (2.2) and (2.3), we have

$$ {}_0^R D_t^{\alpha-\epsilon}(y(t)-y(0)) - {}_0^R D_t^{\alpha}(z(t)-z(0)) = f(t,y(t),y(t-\tau)) - f(t,z(t),z(t-\tau)). $$

It follows from Lemma 2.2 that

$$ \begin{aligned} {}_0^R D_t^{\alpha}(y(t)-z(t)) =& {}_0^R D_t^{\alpha}(y(t)-y(0)) - {}_0^R D_t^{\alpha}(z(t)-z(0)) \\ =& f(t,y(t),y(t-\tau)) - f(t,z(t),z(t-\tau)) + \mathcal{O}(\epsilon){}_0^R D_t^{\alpha-\epsilon}(y(t)-y(0)). \end{aligned} $$

By using the notation I^α for the inverse of the fractional differential operator, the above formula can be written as

$$ y(t)-z(t) = I^\alpha\big(f(t,y(t),y(t-\tau)) - f(t,z(t),z(t-\tau))\big) + \mathcal{O}(\epsilon) I^\epsilon (y(t)-y(0)). $$

It follows from (2.1) that the integral inequality

$$ \begin{aligned} \max_{0\le\xi\le t}|y(\xi)-z(\xi)| \le& \frac{1}{\Gamma(\alpha)} \max_{0\le\xi\le t} \int_0^\xi (\xi-s)^{\alpha-1}\big[\gamma|y(s)-z(s)| \\ &+ \beta|y(s-\tau)-z(s-\tau)|\big]ds + M\epsilon \\ \le& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\gamma \max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)| \\ &+ \beta \max_{0\le\zeta\le s}|y(\zeta-\tau)-z(\zeta-\tau)|\Big]ds + M\epsilon \\ \le& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\gamma \max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)| \\ &+ \beta \max\Big\{\max_{-\tau\le\zeta\le 0}|y(\zeta)-z(\zeta)|, \max_{0\le\zeta\le s-\tau}|y(\zeta)-z(\zeta)|\Big\}\Big]ds + M\epsilon \\ \le& \frac{(\gamma+\beta)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)|ds + M\epsilon, \end{aligned} $$

where M is a constant such that

$$ \max_{0\le t\le T} I^\epsilon(y(t)-y(0))\mathcal{O}(\epsilon) \le M\epsilon. $$

Then, by means of Lemma 2.1, we have

$$ |y(t)-z(t)| \le \max_{0\le\xi\le t}|y(\xi)-z(\xi)| \le M\epsilon E_\alpha((\gamma+\beta)t^\alpha) \le M\epsilon\|E_\alpha((\gamma+\beta)t^\alpha)\|_\infty. \tag{2.4} $$

In view of the convergence of Mittag-Leffler functions, this implies that

$$ \|y(t)-z(t)\|_\infty = \mathcal{O}(\epsilon). \tag{2.5} $$

The proof is complete. □

If the initial function $\varphi(t)$ is inaccurate, then, we need to discuss the dependence of the solution on the initial function, i.e., the influence of the perturbed initial function on the solution.

Theorem 2.2. *Assume that Problem* (1.5) *satisfies* $0 < \alpha < 1$ *and the Lipschitz condition* (2.1). *If* $y(t)$ *and* $z(t)$ *are the uniquely determined solutions of the following problems*

$$\begin{cases} {}_0^R D_t^\alpha (y(t) - y(0)) = f(t, y_t), & t \in [0, T], \\ y(t) = \varphi_1(t), & t \in [-\tau, 0], \end{cases} \tag{2.6}$$

and

$$\begin{cases} {}_0^R D_t^\alpha (z(t) - z(0)) = f(t, z_t), & t \in [0, T], \\ z(t) = \varphi_2(t), & t \in [-\tau, 0], \end{cases} \tag{2.7}$$

respectively, then we have the relation

$$\|y(t) - z(t)\|_\infty = \mathcal{O}\Big(\max_{-\tau \le \zeta \le 0} |\varphi_1(\zeta) - \varphi_2(\zeta)| \Big).$$

Proof. From (2.6), (2.7) and (1.6), we have

$$\begin{aligned} \max_{0 \le \xi \le t} |y(\xi) - z(\xi)| \le & |\varphi_1(0) - \varphi_2(0)| + \max_{0 \le \xi \le t} |I^\alpha (f(\xi, y(\xi), y(\xi - \tau)) - f(\xi, z(\xi), z(\xi - \tau)))| \\ \le & |\varphi_1(0) - \varphi_2(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma \max_{0 \le \zeta \le s} |y(\zeta) - z(\zeta)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta \max_{0 \le \zeta \le s} |y(\zeta - \tau) - z(\zeta - \tau)| ds \\ \le & |\varphi_1(0) - \varphi_2(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma \max_{0 \le \zeta \le s} |y(\zeta) - z(\zeta)| ds + \frac{1}{\Gamma(\alpha)} \cdot \\ & \int_0^t (t-s)^{\alpha-1} \beta \max \Big\{ \max_{-\tau \le \zeta \le 0} |\varphi_1(\zeta) - \varphi_2(\zeta)|, \max_{0 \le \zeta \le s-\tau} |y(\zeta) - z(\zeta)| \Big\} ds. \end{aligned}$$

Obviously, if

$$\max_{0 \le \zeta \le s-\tau} |y(\zeta) - z(\zeta)| < \max_{-\tau \le \zeta \le 0} |\varphi_1(\zeta) - \varphi_2(\zeta)|, \quad \forall s \in [0, t],$$

then

$$\begin{aligned} \max_{0 \le \xi \le t} |y(\xi) - z(\xi)| \le & \Big(1 + \frac{\beta t^\alpha}{\Gamma(\alpha+1)}\Big) \max_{-\tau \le \zeta \le 0} |\varphi_1(\zeta) - \varphi_2(\zeta)| \\ & + \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \max_{0 \le \zeta \le s} |y(\zeta) - z(\zeta)| ds. \end{aligned} \tag{2.8}$$

Furthermore, if

$$\max_{0 \le \zeta \le s-\tau} |y(\zeta) - z(\zeta)| > \max_{-\tau \le \zeta \le 0} |\varphi_1(\zeta) - \varphi_2(\zeta)|, \quad \forall s \in [0, t],$$

then we have

$$\begin{aligned} \max_{0 \le \xi \le t} |y(\xi) - z(\xi)| \le & \max_{-\tau \le \zeta \le 0} |\varphi_1(\zeta) - \varphi_2(\zeta)| \\ & + \frac{\beta + \gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \max_{0 \le \zeta \le s} |y(\zeta) - z(\zeta)| ds. \end{aligned} \tag{2.9}$$

Moreover, if there exists $\eta \in [0,t]$ such that

$$\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)| = \max_{0\le\zeta\le\eta-\tau}|y(\zeta)-z(\zeta)|,$$

and noting that the function $\max_{0\le\zeta\le s-\tau}|y(\zeta)-z(\zeta)|$ is nondecreasing, then

$$\begin{aligned}
&\max_{0\le\xi\le t}|y(\xi)-z(\xi)| \\
\le&\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)| + \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)|ds \\
&+\frac{\beta}{\Gamma(\alpha)}\int_0^\eta (t-s)^{\alpha-1}\max_{0\le\zeta\le s-\tau}|y(\zeta)-z(\zeta)|ds \\
&+\frac{\beta}{\Gamma(\alpha)}\int_\eta^t (t-s)^{\alpha-1}\max_{0\le\zeta\le s-\tau}|y(\zeta)-z(\zeta)|ds \\
\le&\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)| + \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)|ds \\
&+\frac{\beta}{\Gamma(\alpha)}\int_0^\eta (t-s)^{\alpha-1}\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)|ds \\
&+\frac{\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)|ds \\
\le&\Big(1+\frac{\beta t^\alpha}{\Gamma(\alpha+1)}\Big)\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)| \\
&+\frac{\beta+\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\max_{0\le\zeta\le s}|y(\zeta)-z(\zeta)|ds.
\end{aligned} \tag{2.10}$$

Thus it follows from (2.8), (2.9), (2.10) and Lemma 2.1 that

$$\begin{aligned}
|y(t)-z(t)| \le& \max_{0\le\xi\le t}|y(\xi)-z(\xi)| \\
\le&\Big(1+\frac{\beta t^\alpha}{\Gamma(\alpha+1)}\Big)\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)|E_\alpha((\gamma+\beta)t^\alpha) \\
\le&\Big(1+\frac{\beta T^\alpha}{\Gamma(\alpha+1)}\Big)\|E_\alpha((\gamma+\beta)t^\alpha)\|_\infty\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)|.
\end{aligned} \tag{2.11}$$

In view of the convergence of Mittag-Leffler functions, we have

$$\|y(t)-z(t)\|_\infty = \mathcal{O}\Big(\max_{-\tau\le\zeta\le 0}|\varphi_1(\zeta)-\varphi_2(\zeta)|\Big).$$

The proof is complete. □

Moreover, if the right-hand side function f is inaccurate, then we need to discuss the dependence of the solution on the perturbed right-hand side function.

Theorem 2.3. *Assume that Problem* (1.5) *satisfies* $0 < \alpha < 1$ *and the Lipschitz condition* (2.1). *If* $y(t)$ *and* $z(t)$ *are the uniquely determined solutions of the following problems*

$$\begin{cases} {}_0^R D_t^\alpha (y(t) - y(0)) = f(t, y_t), & t \in [0, T], \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{2.12}$$

and

$$\begin{cases} {}_0^R D_t^\alpha (z(t) - z(0)) = \tilde{f}(t, z_t), & t \in [0, T], \\ z(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{2.13}$$

respectively, then we have the relation

$$\|y(t) - z(t)\|_\infty = \mathcal{O}(\|f - \tilde{f}\|_\infty).$$

Proof. From (2.12) and (2.13), we easily obtain the relationship

$$\begin{aligned} \max_{0 \le \xi \le t} |y(\xi) - z(\xi)| \le & \max_{0 \le \xi \le t} |I^\alpha (f(\xi, y(\xi), y(\xi - \tau)) - \tilde{f}(\xi, z(\xi), z(\xi - \tau)))| \\ = & \max_{0 \le \xi \le t} |I^\alpha (f(\xi, y(\xi), y(\xi - \tau)) - f(\xi, z(\xi), z(\xi - \tau)) \\ & + f(\xi, z(\xi), z(\xi - \tau)) - \tilde{f}(\xi, z(\xi), z(\xi - \tau)))| \\ \le & \frac{\gamma + \beta}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \max_{0 \le \zeta \le s} |y(\zeta) - z(\zeta)| ds + \frac{t^\alpha}{\Gamma(\alpha + 1)} \|f - \tilde{f}\|_\infty. \end{aligned} \tag{2.14}$$

By means of Lemma 2.1 and noting that the convergence of Mittag-Leffler functions, we have

$$\|y(t) - z(t)\|_\infty = \mathcal{O}(\|f - \tilde{f}\|_\infty).$$

So, we complete the proof. □

Finally, we estimate the solution of the Problem (1.5), as shown in the following theorem.

Theorem 2.4. *Assume that for Problem* (1.5), $0 < \alpha < 1$, *and there exist constants* $M > 0$, $\beta > 0$, *such that* f *satisfies*

$$\|f(t, x(t), x(t - \tau))\|_\infty \le M,$$

or

$$\max_{0 \le s \le t} |f(s, x(s), x(s - \tau))| \le \beta \max\Big\{ \max_{-\tau \le s \le 0} |\varphi(s)|, \max_{0 \le s \le t} |x(s)| \Big\}, \quad \forall t \in [0, T].$$

Then the solution $x(t)$ *of Problem* (1.5) *satisfies*

$$|x(t)| \le |\varphi(0)| + \frac{M t^\alpha}{\Gamma(\alpha + 1)},$$

or

$$|x(t)| \le \max_{0 \le s \le t} |x(s)| \le L(t) E_\alpha(\beta t^\alpha), \quad \forall t \in [0, T],$$

where

$$L(t) = \max_{-\tau \le s \le 0} |\varphi(s)| \Big(1 + \frac{\beta t^\alpha}{\Gamma(\alpha + 1)}\Big).$$

Proof. If

$$\|f(t,x(t),x(t-\tau))\|_\infty \le M,$$

it follows from (1.6) that

$$\begin{aligned} |x(t)| \le& |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \max_{0\le\xi\le t} |f(\xi,x(\xi),x(\xi-\tau))|ds \\ \le& |\varphi(0)| + \frac{Mt^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{2.15}$$

Obviously, the conclusion is right. On the other hand, according to (1.6), we can easily obtain that

$$|x(t)| \le |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s),x(s-\tau))|ds. \tag{2.16}$$

Then

$$\begin{aligned} \max_{0\le\xi\le t} |x(\xi)| \le& |\varphi(0)| + \max_{0\le\xi\le t} \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1} |f(s,x(s),x(s-\tau))|ds \\ \le& |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \max_{0\le\zeta\le s} |f(\zeta,x(\zeta),x(\zeta-\tau))|ds \\ \le& |\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta \max\Big\{\max_{-\tau\le\zeta\le 0} |\varphi(\zeta)|, \max_{0\le\zeta\le s} |x(\zeta)|\Big\} ds. \end{aligned} \tag{2.17}$$

By means of Lemma 2.1, the conclusion can be obtained by a similar proof process of Theorem 2.2. The proof is complete. ☐

Remark 2.1. For the following initial value problem of a linear FDDE in the form

$$\begin{cases} {}_0^C D_t^\alpha y(t) = a(t)y(t) + b(t)y(t-\tau), & t \in [0,T], \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{2.18}$$

where $\varphi(t), a(t), b(t)$ are the given continuous functions, $0 < \alpha < 1$. Obviously

$$\begin{aligned} \max_{0\le s\le t} |a(s)y(s) + b(s)y(s-\tau)| \le& \max_{0\le s\le t} \{|a(s)||y(s)| + |b(s)||y(s-\tau)|\} \\ \le& \max\{\|a\|_\infty, \|b\|_\infty\} \Big(\max_{0\le s\le t} |y(s)| + \max_{0\le s\le t} |y(s-\tau)| \Big) \\ \le& 2\max\{\|a\|_\infty, \|b\|_\infty\} \max\Big\{ \max_{-\tau\le s\le 0} |\varphi(s)|, \max_{0\le s\le t} |y(s)| \Big\}. \end{aligned}$$

Then the solution $y(t)$ of the Problem (2.18) satisfies

$$|y(t)| \le \max_{0\le s\le t} |y(s)| \le L(t) E_\alpha(\beta t^\alpha), \quad \forall t \in [0,T],$$

where

$$L(t) = \max_{-\tau\le s\le 0} |\varphi(s)| \Big(1 + \frac{\beta t^\alpha}{\Gamma(\alpha+1)}\Big) \quad \text{and} \quad \beta = 2\max\{\|a\|_\infty, \|b\|_\infty\}.$$

3 Numerical examples

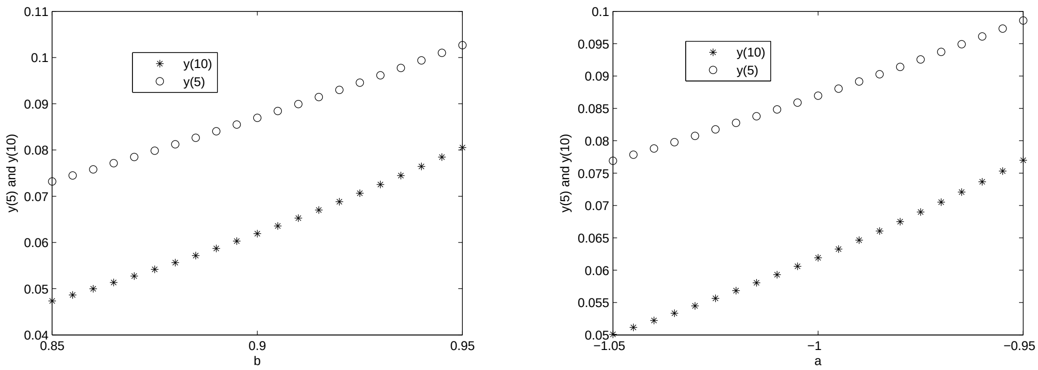
In this section, we consider the FDDE of the form

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = a y(t) + b y(t-\tau), & t \in [0, T], \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{3.1}$$

where $T = 10$, $\tau = 1.0$. We always apply 2-fractional linear multi-step methods (2-FLMMs) proposed in [10] to obtain approximations to $y(10)$ in the following simulations.

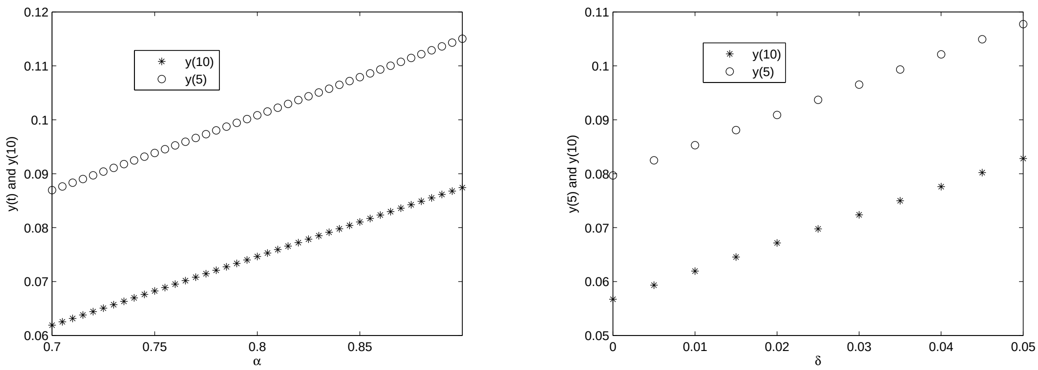
First, we take $a = -1$, $b = 0.9$, $\varphi(t) = t^2$, and investigate the perturbation values of $y(5)$, $y(10)$ when α varies from 0.7. The obtained numerical results are given in Table 1. For the fixed step size $h = 0.05$, the numerical solutions give the approximately-linear relationship between α and $y(5)$, $y(10)$ in Fig. 1(a).

Second, we investigate the dependence of the solution on the parameters a and b, i.e., on the perturbed right-hand function of the FDDE (3.1). If we take $\alpha = 0.7$, $\varphi(t) =$



(a) Relationships between α and $y(5)$, $y(10)$ (b) Relationships between a and $y(5)$, $y(10)$

Figure 1: Numerical solutions for Eq. (3.1) with the effect for the parameters α and a.



(a) Relationships between b and $y(5)$, $y(10)$ (b) Relationships between δ and $y(5)$, $y(10)$

Figure 2: Numerical solutions for Eq. (3.1) with the effect for the parameters b and δ.

Table 1: Approximate values of $y(10)$ for $h = 0.1, 0.05, 0.01$ and the perturbed α.

α	h=0.1	h=0.05	h=0.001
0.700	6.6479e-002	6.1908e-002	5.8422e-002
0.705	6.7156e-002	6.2539e-002	5.9018e-002
0.710	6.7835e-002	6.3171e-002	5.9614e-002
0.715	6.8514e-002	6.3804e-002	6.0211e-002
0.720	6.9194e-002	6.4437e-002	6.0809e-002
0.725	6.9875e-002	6.507e-002	6.1407e-002
0.730	7.0557e-002	6.5707e-002	6.2007e-002
0.735	7.1239e-002	6.6343e-002	6.2607e-002
0.740	7.1923e-002	6.6980e-002	6.3208e-002
0.745	7.2607e-002	6.7617e-002	6.3809e-002
0.750	7.3291e-002	6.8255e-002	6.4411e-002

Table 2: Approximate values of $y(10)$ for $\alpha = 0.7$, $b = 0.9$, $\varphi(t) = t^2$ and the perturbed a.

a	$h = 0.1$	$h = 0.05$	$h = 0.001$
-1.000	6.6479e-002	6.1908e-002	5.8422e-002
-1.005	6.5067e-002	6.0594e-002	5.7184e-002
-1.010	6.3688e-002	5.9311e-002	5.5975e-002
-1.015	6.2343e-002	5.8060e-002	5.4795e-002
-1.020	6.1030e-002	5.6838e-002	5.3643e-002
-1.025	5.9748e-002	5.5646e-002	5.2519e-002
-1.030	5.8497e-002	5.4482e-002	5.1421e-002
-1.035	5.7276e-002	5.3345e-002	5.0350e-002
-1.040	5.6084e-002	5.2236e-002	4.9304e-002
-1.045	5.4920e-002	5.1153e-002	4.8282e-002
-1.050	5.3783e-002	5.0095e-002	4.7285e-002

Table 3: Approximate values of $y(10)$ for $\alpha = 0.7$, $a = -1.0$, $\varphi(t) = t^2$ and the perturbed b.

b	$h = 0.1$	$h = 0.05$	$h = 0.001$
0.900	6.6479e-002	6.1908e-002	5.8422e-002
0.905	6.8268e-002	6.3572e-002	5.9991e-002
0.910	7.0102e-002	6.52783e-002	6.1600e-002
0.915	7.1980e-002	6.70263e-002	6.3247e-002
0.920	7.3906e-002	6.88174e-002	6.4936e-002
0.925	7.5879e-002	7.0652e-002	6.6666e-002
0.930	7.7900e-002	7.25333e-002	6.8439e-002
0.935	7.9972e-002	7.4460e-002	7.0255e-002
0.940	8.2094e-002	7.64338e-002	7.2116e-002
0.945	8.4268e-002	7.84559e-002	7.4022e-002
0.950	8.6495e-002	8.05273e-002	7.5974e-002

t^2, we calculate the approximate values of $y(10)$ by using the step sizes $h = 0.1$, 0.05, 0.001, while the value a varies from -1.05 to -1.0, or the value b varies from 0.90 to 0.95. The results are shown in Tables 2 and 3 respectively. In Figs. 1(b) and 2(a), the approximately-linear relationships between a and $y(5)$, $y(10)$, b and $y(5)$, $y(10)$ are

Table 4: Approximate values of $y(10)$ for $\alpha = 0.7$, $a = -1$, $b = 0.9$ and the perturbed $\varphi(t)$.

δ	$h = 0.1$	$h = 0.05$	$h = 0.001$
0.000	6.6479e-002	6.1908e-002	5.8422e-002
0.005	7.0022e-002	6.5422e-002	6.1914e-002
0.010	7.3565e-002	6.8936e-002	6.5405e-002
0.015	7.7108e-002	7.2451e-002	6.8897e-002
0.020	8.0650e-002	7.5965e-002	7.2388e-002
0.025	8.4193e-002	7.9479e-002	7.5880e-002
0.030	8.7736e-002	8.2993e-002	7.9371e-002
0.035	9.1278e-002	8.6507e-002	8.2862e-002
0.040	9.4821e-002	9.0021e-002	8.6354e-002
0.045	9.8364e-002	9.3536e-002	8.9845e-002
0.050	1.0190e-001	9.7050e-002	9.3337e-002

given respectively with the fixed the step size $h = 0.05$.

Finally, we suppose

$$y(t) = \varphi(t) + \delta, \quad \forall t \in [-\tau, 0],$$

and investigate the dependence of the solution on the parameter δ, i.e., on the perturbed initial function $\varphi(t)$ of the Problem (3.1). In Table 4, we present the values of $y(10)$, based on the fixed values $a = -1.0$, $b = 0.9$, $\alpha = 0.7$, while the value of δ varies from 0 to 0.05. The approximately-linear relationships between δ and $y(5), y(10)$ are shown in Fig. 2(b).

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