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The Lower Bounds of Eigenvalues by the Wilson Element in Any Dimension

Youai Li*

College of Computer and Information Engineering, Beijing Technology and Business University, Beijing 10080, China

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Abstract. In this paper, we analyze the Wilson element method of the eigenvalue problem in arbitrary dimensions by combining a new technique recently developed in [10] and the a posteriori error result. We prove that the discrete eigenvalues are smaller than the exact ones.

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1 The Wilson element in any dimension

This paper is devoted to the finite element approximation of the following elliptic eigenvalue problem: find $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ with

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \lambda(\rho u, v)_{L^2(\Omega)}, \text{ for any } v \in H_0^1(\Omega), \text{ with } \big\|\rho^{\frac{1}{2}} u\big\|_{L^2(\Omega)} = 1, \tag{1.1}$$

where $\rho \in L^\infty(\Omega)$ is a positive function.

Let $\mathcal{T}_h$ be a regular n-rectangular triangulation of the n-rectangular domain $\Omega \subset \mathbb{R}^n$ with $2 \leq n$ in the sense that

$$\bigcup_{K \in \mathcal{T}_h} K = \bar{\Omega},$$

two distinct elements K and K' in $\mathcal{T}_h$ are either disjoint, or share the ℓ-dimensional hyper-plane, $\ell = 0, \cdots, n-1$. Let $\mathcal{H}_h$ denote the set of all $n-1$ dimensional hyper-planes in $\mathcal{T}_h$ with the set of interior $n-1$ dimensional hyper-planes $\mathcal{H}_h(\Omega)$ and the set of boundary $n-1$ dimensional hyper-planes $\mathcal{H}_h(\partial\Omega)$. We let $\mathcal{N}_h$ denote the set of nodes of $\mathcal{T}_h$ with the set of internal nodes $\mathcal{N}_h(\Omega)$ and the set of boundary nodes $\mathcal{N}_h(\partial\Omega)$.

*Corresponding author.
Email: liya@th.btbu.edu.cn (Y. A. Li)

For each $K \in \mathcal{T}_h$, we introduce the following affine invertible transformation

$$F_K: \hat{K} \to K, \ \ x_i = h_{x_i,K}\xi_i + x_i^0,$$

with the center $(x_1^0, x_2^0, \cdots, x_n^0)$ and the lengths $2h_{x_i,K}$ of K in the directions of the x_i-axis, and the reference element $\hat{K} = [-1,1]^n$. In this paper, we only consider the uniform mesh with $h_{x_i} = h_{x_i,K}$ for any $K \in \mathcal{T}_h$. In addition, we set $h = \max_{1\leq i\leq n} h_{x_i}$.

Denote by $Q_{nD}(\hat{K})$ the nonconforming Wilson element space [17] on the reference element defined by

$$Q_{nD}(\hat{K}) = Q_1(\hat{K}) + \text{span}\left\{\xi_1^2 - 1, \xi_2^2 - 1, \cdots, \xi_n^2 - 1\right\}, \tag{1.2}$$

where $Q_1(\hat{K})$ is the space of polynomials of degree≤ 1 in each variable. The nonconforming Wilson element space V_h^{nc} is then defined as

$$V_h^{nc} := \Big\{ v \in L^2(\Omega): \ v|_K \circ F_K \in Q_{nD}(\hat{K}) \text{ for each } K \in \mathcal{T}_h, \ v \text{ is continuous at the internal nodes, and vanishes at the boundary nodes} \Big\}.$$

Define the discrete semi-norm on V_h^{nc} by

$$|v|_h^2 = \sum_{K\in\mathcal{T}_h} \|\nabla v\|^2_{L^2(K)}.$$

By the Poincare inequality, we have $|\cdot|_h$ as a norm on V_h^{nc}. The finite element approximation of Problem (1.1) reads: find $(\lambda_h, u_h) \in \mathbb{R} \times V_h^{nc}$, such that

$$(\nabla_h u_h, \nabla_h v_h)_{L^2(\Omega)} = \lambda_h(\rho u_h, v_h)_{L^2(\Omega)}, \text{ for any } v_h \in V_h^{nc}, \text{ with } \big\|\rho^{\frac{1}{2}} u_h\big\|_{L^2(\Omega)} = 1. \tag{1.3}$$

The purpose of this paper is to analyze the lower approximation property of eigenvalues produced in (1.3). By combining the method based on the identity from [1,11] and the technique developed for the Adini element in a recent paper [10], we prove that the discrete eigenvalues are smaller than the exact ones when the meshsize h is small enough. Compared to the result of [19] only for the three dimensions, the novelties of the paper are of twofold: It analyzes the Wilson element in any dimension [17]; it is able to weaken the regularity condition on the eigenfunction.

The rest of the paper is organized as follows. In the following section, we prove the main result of this paper, ie., the discrete eigenvalues produced by the Wilson element are smaller than the exact ones. Some proof details are presented in Section 3.

2 Lower approximations of eigenvalues

We show that the approximate eigenvalues are smaller than the exact ones in this section. We first define the canonical interpolation. Let

$$a_i = (\xi_{1i}, \xi_{2i}, \cdots, \xi_{ni}), \qquad i = 1, \cdots, 2^n,$$

denote the vertexes of the n-square $[-1,1]^n$. Given $v \in H^2(K)$, the interpolation $\Pi_K v$ is defined by

$$\Pi_K v = \frac{1}{2^n}\sum_{j=1}^{2^n}\prod_{k=1}^{n} v(P_j)(1+\xi_{kj}\xi_k)\circ F_K^{-1} + \sum_{k=1}^{n}\frac{h_{x_k,K}^2}{2}\Pi_K^0\frac{\partial^2 v}{\partial x_k^2}(\xi_k^2-1)\circ F_K^{-1}, \tag{2.1}$$

where P_j, $j=1,\cdots,2^n$, denote the vertexes of K, and $\Pi_K^0 w$ denotes the integral average of w over K.

Given $v \in H_0^1(\Omega)\cap H^2(\Omega)$, the interpolation $v_I \in V_h^{nc}$ is defined by

$$v_I|_K = \Pi_K v|_K, \quad \text{for any} \quad K \in \mathcal{T}_h. \tag{2.2}$$

For this interpolation, we have the following error estimate:

Lemma 2.1. *Let $v \in H^{2+s}(K)$ with $0 < s \leq 1$. Then*

$$\|v-\Pi_K v\|_{L^2(K)} + h\|\nabla(v-\Pi_K v)\|_{L^2(K)} \lesssim h^{2+s}|v|_{H^{2+s}(K)}. \tag{2.3}$$

We will make use of error estimates for the approximation of eigenvalue problems by the Wilson element method. These estimates follow from the general theory obtained in [2,3,7] and the properties of the Wilson element [16–18]. In particular, it is known that

$$|u-u_h|_h \lesssim h \quad \text{and} \quad \|u-u_h\|_{L^2(\Omega)} \lesssim h^2, \tag{2.4}$$

provided that the eigenfunction $u \in H^2(\Omega)$.

Theorem 2.1. *Let (λ, u) and (λ_h, u_h) be the solutions of problems* (1.1) *and* (1.3)*, respectively, and the term $T_P > 0$ be defined as in* (3.36) *in the next section. Assume that*

$$h^2 \lesssim |u-u_h|_h^2 + T_P \quad \textit{and} \quad u \in H_0^1(\Omega)\cap H^{2+s}(\Omega),$$

with $0 < s \leq 1$. Then,

$$\lambda_h \leq \lambda, \tag{2.5}$$

provided that h is small enough.

Proof. We shall follow the idea of [1,11] to use some identity for the error of the eigenvalue. Such an identity can be actually found in [1,11,20]. We derive it herein only for readers' convenience. By the eigenvalue problem (1.1) and the discrete eigenvalue problem (1.3), we derive as

$$\begin{aligned}\lambda + \lambda_h &= |u|_h^2 + |u_h|_h^2 = |u-u_h|_h^2 + 2(\nabla u, \nabla_h u_h)_{L^2(\Omega)}\\ &= |u-u_h|_h^2 + 2(\nabla_h u_I, \nabla_h u_h)_{L^2(\Omega)} + 2(\nabla_h(u-u_I), \nabla_h u_h)_{L^2(\Omega)}\\ &= |u-u_h|_h^2 + 2\lambda_h(\rho u_I, u_h)_{L^2(\Omega)} + 2(\nabla_h(u-u_I), \nabla_h u_h)_{L^2(\Omega)}.\end{aligned} \tag{2.6}$$

This leads to

$$\begin{aligned}\lambda-\lambda_h &= |u-u_h|_h^2+2\lambda_h\big(\rho(u_I-u_h),u_h\big)_{L^2(\Omega)}+2\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)}\\ &= |u-u_h|_h^2-\lambda_h\big(\rho(u_I-u_h),u_I-u_h\big)_{L^2(\Omega)}\\ &\quad+\lambda_h\big(\rho(u_I-u_h),u_I+u_h\big)_{L^2(\Omega)}+2\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)}\\ &= |u-u_h|_h^2-\lambda_h\big(\rho(u_I-u_h),u_I-u_h\big)_{L^2(\Omega)}\\ &\quad+\lambda_h\big(\|\rho^{\frac12}u_I\|^2_{L^2(\Omega)}-\|\rho^{\frac12}u\|^2_{L^2(\Omega)}\big)+2\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)}. \end{aligned} \tag{2.7}$$

By the error estimates of the interpolation u_I and the finite element solution u_h of the eigenfunction u in (2.3) and (2.4), respectively, we can bound the second and third terms as following

$$\big(\rho(u_I-u_h),u_I-u_h\big)\lesssim h^4, \tag{2.8a}$$

$$\Big|\big\|\rho^{\frac12}u_I\big\|^2_{L^2(\Omega}-\big\|\rho^{\frac12}u\big\|^2_{L^2(\Omega)}\Big|\lesssim h^{2+s}. \tag{2.8b}$$

The fourth term will be analyzed in the next section, see (3.36), where it will be proved that

$$2\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)}=T_p+\mathcal{O}(h^{2+s}),\quad \text{with } T_P>0. \tag{2.9}$$

Under the assumption

$$h^2\lesssim|u-u_h|_h^2,$$

we conclude that $\lambda_h\le\lambda$ when the meshsize h is small enough. □

3 Expansion of $\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)}$

In this section, we shall analyze the key term $(\nabla_h(u-u_I),\nabla_h u_h)_{L^2(\Omega)}$. We will use the idea that combines the technique developed in [10] herein and the a posteriori error result from [12].

For simplicity of presentation and notation, we only analyze the case $n=3$. We need some further notation of the mesh. Given any face $f\in\mathcal{F}_h(\Omega)$ with the diameter h_f we assign one fixed unit normal $\nu:=(\nu_1,\nu_2,\nu_3)$. For f on the boundary we choose ν as the unit outward normal to Ω. Once ν has been fixed on f, in relation to ν one defines the elements $K_-\in\mathcal{T}_h$ and $K_+\in\mathcal{T}_h$, with $f=K_+\cap K_-$. Given $f\in\mathcal{F}_h(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω, with $d=1,2$, we denote by

$$[v]:=(v|_{K_+})|_f-(v|_{K_-})|_f,$$

the jump, and

$$\{v\}=\frac{1}{2}\big((v|_{K_+})|_f+(v|_{K_-})|_f\big),$$

the average of v across f. In addition, we let

$$\omega_f = K_+ \cup K_-.$$

Integrating by parts yields

$$\begin{aligned}
&\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)}\\
=&-(u-u_I,\mathrm{div}_h\nabla_h u_h)_{L^2(\Omega)}+\sum_{k\in T_h}\int_{\partial K}(u-u_I)\frac{\partial u_h}{\partial \nu}df\\
=&-(u-u_I,\mathrm{div}_h\nabla_h u_h)_{L^2(\Omega)}+\sum_{f\in\mathcal{F}_h(\Omega)}\int_f\{u-u_I\}\Big[\frac{\partial u_h}{\partial \nu}\Big]df\\
&+\sum_{f\in\mathcal{F}_h(\Omega)}\int_f[u-u_I]\Big\{\frac{\partial u_h}{\partial \nu}\Big\}df+\sum_{f\in\mathcal{F}_h(\partial\Omega)}\int_f(u-u_I)\frac{\partial u_h}{\partial \nu}df\\
=&I_1+I_2+I_3+I_4,
\end{aligned}\tag{3.1}$$

where div_h is the elementwise defined divergence operator. We shall analyze the four terms on the right-hand side of (3.1) one by one.

3.1 The estimates of the first two terms

It follows from the error estimate of the interpolation in (2.3) that

$$|I_1|\leqslant ||u-u_I||_{L^2(\Omega)}||\mathrm{div}_h\nabla_h u_h||_{L^2(\Omega)}\lesssim h^{2+s}|u|^2_{H^{2+s}(\Omega)},\tag{3.2}$$

where we use the fact that

$$\|\,\mathrm{div}_h\,\nabla_h u_h\|_{L^2(\Omega)}\lesssim |u|_{H^2(\Omega)}.$$

To analyze the second term I_2, we need the following result from a posteriori error analysis of the nonconforming element method for the eigenvalue problem [12].

$$\sum_{f\in\mathcal{F}_h(\Omega)}h_f\Big\|\Big[\frac{\partial u_h}{\partial \nu}\Big]\Big\|^2_{L^2(f)}\lesssim |u-u_h|^2_h.\tag{3.3}$$

Then, we use, the Cauchy-Schwarz inequality, the trace theorem, the error estimate of u_I in (2.3), and the error estimate of u_h in (2.4) to derive as

$$\begin{aligned}
|I_2|=&\Big|\sum_{f\in\mathcal{F}_h(\Omega)}\int_f\{u-u_I\}\Big[\frac{\partial u_h}{\partial \nu}\Big]df\Big|\lesssim\sum_{f\in\mathcal{F}_h(\Omega)}||\{u-u_I\}||_{L^2(f)}\Big\|\Big[\frac{\partial u_h}{\partial \nu}\Big]\Big\|_{L^2(f)}\\
\lesssim&\sum_{f\in\mathcal{F}_h(\Omega)}\Big(h^{-\frac12}\|u-u_I\|_{L^2(\omega_f)}+h^{\frac12}\|\nabla_h(u-u_I)\|_{L^2(\omega_f)}\Big)\Big\|\Big[\frac{\partial u_h}{\partial \nu}\Big]\Big\|_{L^2(f)}\\
\lesssim&\sum_{f\in\mathcal{F}_h(\Omega)}h^{1+s+\frac12}|u|_{H^{2+s}(\omega_f)}\Big\|\Big[\frac{\partial u_h}{\partial \nu}\Big]\Big\|_{L^2(f)}\\
\lesssim&\Big(\sum_{f\in\mathcal{F}_h(\Omega)}h^{2+2s}|u|^2_{H^{2+s}(\omega_f)}\Big)^{\frac12}\Big(\sum_{f\in\mathcal{F}_h(\Omega)}h\Big\|\Big[\frac{\partial u_h}{\partial \nu}\Big]\Big\|^2_{L^2(f)}\Big)^{\frac12}\\
\lesssim&h^{1+s}|u|_{H^{2+s}(\Omega)}\|\nabla_h(u-u_h)\|_{L^2(\Omega)}\lesssim h^{2+s}|u|^2_{H^{2+s}(\Omega)}.
\end{aligned}\tag{3.4}$$

3.2 The expansion of the third term

We let $\mathcal{F}_X(\Omega)$ denote the set of internal faces that is perpendicular to the x-axis, $\mathcal{F}_Y(\Omega)$ the set of internal faces that is perpendicular to the Y-axis, and $\mathcal{F}_Z(\Omega)$ the set of internal faces that is perpendicular to the Z-axis. Then, we have the following decomposition for the term I_3 in (3.1)

$$\begin{aligned} I_3 = &\sum_{f\in\mathcal{F}_X(\Omega)} \int_f [u-u_I]\Big\{\frac{\partial u_h}{\partial \nu}\Big\}df + \sum_{f\in\mathcal{F}_Y(\Omega)} \int_f [u-u_I]\Big\{\frac{\partial u_h}{\partial \nu}\Big\}df \\ &+ \sum_{f\in\mathcal{F}_Z(\Omega)} \int_f [u-u_I]\Big\{\frac{\partial u_h}{\partial \nu}\Big\}df. \end{aligned} \tag{3.5}$$

Since the expansions of the three terms on the right-hand side of (3.5) are similar, we only study the first term. Given $f\in\mathcal{F}_X(\Omega)$ with $f=K_1\cap K_2$, let (x_{0,K_1},y_0,z_0) and (x_{0,K_2},y_0,z_0) be the centers of K_1 and K_2, respectively. In this case we can fix the unit normal vector ν such that it agrees with the positive direction of the x-axis. Without loss of generality, we assume that $x_{0,K_1}<x_{0,K_2}$. Therefore, we have

$$\begin{aligned} &\int_f [u-u_I]\Big\{\frac{\partial u_h}{\partial \nu}\Big\}df \\ =&\int_f \big((u-u_I)|_{K_1}-(u-u_I)|_{K_2}\big)\Big\{\frac{\partial u_h}{\partial x}\Big\}df \\ =&-\int_f \big(u_I|_{K_1}-u_I|_{K_2}\big)\Big\{\frac{\partial u_h}{\partial x}-\frac{\partial u}{\partial x}\Big\}df-\int_f \big(u_I|_{K_1}-u_I|_{K_2}\big)\frac{\partial u}{\partial x}df \\ =&S_{1,f}+S_{2,f}. \end{aligned} \tag{3.6}$$

Let $\Pi^0_{K_1}(\partial^2 u/\partial y^2)$ (resp. $\Pi^0_{K_1}(\partial^2 u/\partial z^2)$) and $\Pi^0_{K_2}(\partial^2 u/\partial y^2)$ (resp. $\Pi^0_{K_2}(\partial^2 u/\partial z^2)$) denote the integral averages of $\partial^2 u/\partial y^2$ (resp. $\partial^2 u/\partial z^2$) on K_1 and K_2, respectively. It follows from the definition of the interpolation u_I that only the last two terms in (2.1) are possibly discontinuous across the face f. Therefore, we have

$$\begin{aligned} S_{2,f}=&-\frac{1}{2}\int_f\Big(\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2}-\Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}\Big)\big((y-y_0)^2-h_y^2\big)\frac{\partial u(x_m,y,z)}{\partial x}dydz \\ &-\frac{1}{2}\int_f\Big(\Pi^0_{K_1}\frac{\partial^2 u}{\partial z^2}-\Pi^0_{K_2}\frac{\partial^2 u}{\partial z^2}\Big)\big((z-z_0)^2-h_z^2\big)\frac{\partial u(x_m,y,z)}{\partial x}dydz \\ =&S_{3,f}+S_{4,f}, \end{aligned} \tag{3.7}$$

with $x_m=(x_{0,K_1}+x_{0,K_2})/2$ and $(x_m,y,z)\in f$. Since the expansions of both terms $S_{3,f}$ and $S_{4,f}$ can be done in a similar way, we only consider the former. Since the average of $((y-y_0)^2-h_y^2)$ over f is equal to

$$H_0=\frac{1}{2h_y}\int_{y_0-h_y}^{y_0+h_y}((y-y_0)^2-h_y^2)dy=\frac{1}{2h_y}\Big(\frac{1}{3}(y-y_0)^3-h_y^2y\Big)\Big|_{y_0-h_y}^{y_0+h_y}=-\frac{2}{3}h_y^2, \tag{3.8}$$

we obtain the following decomposition of $S_{3,f}$ by adding and subtracting the average $-2h_y^2/3$:

$$\begin{aligned}S_{3,f} =& -\frac{1}{2}\int_f \Big(\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2} - \Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}\Big)\Big((y-y_0)^2 - \frac{h_y^2}{3}\Big)\frac{\partial u(x_m,y,z)}{\partial x}dydz \\ &+ \frac{h_y^2}{3}\int_f \Big(\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2} - \Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}\Big)\frac{\partial u(x_m,y,z)}{\partial x}dydz \\ =& J_{1,f} + S_{5,f}. \end{aligned} \tag{3.9}$$

In what follows we analyze the term $S_{5,f}$. Adding and substracting the term $\partial u(x_m,p,q)/\partial x$, we use the definitions of the projection operators $\Pi^0_{K_1}$ and $\Pi^0_{K_2}$ to proceed as

$$\begin{aligned} & 8h_xh_yh_z\Big(\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2} - \Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}\Big)\frac{\partial u(x_m,y,z)}{\partial x} \\ =& \int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Big(\int_{x_{0,K_1}-h_x}^{x_{0,K_1}+h_x}\frac{\partial^2 u(t,p,q)}{\partial p^2}dt - \int_{x_{0,K_2}-h_x}^{x_{0,K_2}+h_x}\frac{\partial^2 u(t,p,q)}{\partial p^2}dt\Big)dpdq\frac{\partial u(x_m,y,z)}{\partial x} \\ =& \int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\Big(\frac{\partial^2 u(x_{0,K_1}+t,p,q)}{\partial p^2} - \frac{\partial^2 u(x_{0,K_2}+t,p,q)}{\partial p^2}\Big)\frac{\partial u(x_m,y,z)}{\partial x}dtdpdq \\ =& \int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\Big(\frac{\partial^2 u(x_{0,K_1}+t,p,q)}{\partial p^2} - \frac{\partial^2 u(x_{0,K_2}+t,p,q)}{\partial p^2}\Big) \\ &\quad\times\Big(\frac{\partial u(x_m,y,z)}{\partial x} - \frac{\partial u(x_m,p,q)}{\partial x}\Big)dtdpdq \\ &+ \int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\Big(\frac{\partial^2 u(x_{0,K_1}+t,p,q)}{\partial p^2} - \frac{\partial^2 u(x_{0,K_2}+t,p,q)}{\partial p^2}\Big)\frac{\partial u(x_m,p,q)}{\partial x}dtdpdq \\ =& 8h_xh_yh_zJ'_{2,f} + S_{6,f}. \end{aligned} \tag{3.10}$$

Integrating by parts, and adding and subtracting the term $-\partial^2 u(x+t,p,q)/\partial x\partial p$, this gives

$$\begin{aligned} S_{6,f} =& -\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\Big(\frac{\partial u(x_{0,K_1}+t,p,q)}{\partial p} - \frac{\partial u(x_{0,K_2}+t,p,q)}{\partial p}\Big)\frac{\partial^2 u(x_m,p,q)}{\partial p\partial x}dtdpdq \\ &+ \int_{z_0-h_z}^{z_0+h_z}\int_{-h_x}^{h_x}\Big(\frac{\partial u(x_{0,K_1}+t,p,q)}{\partial p} - \frac{\partial u(x_{0,K_2}+t,p,q)}{\partial p}\Big)\frac{\partial u(x_m,p,q)}{\partial x}\Big|_{y_0-h_y}^{y_0+h_y}dtdq \\ =& \int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\int_{x_{0,K_1}}^{x_{0,K_2}}\frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}\Big(\frac{\partial^2 u(x_m,p,q)}{\partial x\partial p} - \frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}\Big)dxdtdpdq \\ &+ \int_{z_0-h_z}^{z_0+h_z}\int_{-h_x}^{h_x}\Big(\frac{\partial u(x_{0,K_1}+t,p,q)}{\partial p} - \frac{\partial u(x_{0,K_2}+t,p,q)}{\partial p}\Big)\frac{\partial u(x_m,p,q)}{\partial x}\Big|_{y_0-h_y}^{y_0+h_y}dtdq \\ &+ \int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\int_{x_{0,K_1}}^{x_{0,K_2}}\frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}\frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}dxdtdpdq \\ =& 8h_xh_yh_z(J'_{3,f} + J'_{4,f} + J'_{5,f}). \end{aligned} \tag{3.11}$$

Let

$$J_{\ell,f} = \int_f J'_{\ell,f}dydz, \qquad \ell = 2,\cdots,5.$$

Therefore, we get

$$S_{5,f}=\frac{h_y^2}{3}(J_{2,f}+J_{3,f}+J_{4,f}+J_{5,f}). \tag{3.12}$$

3.3 The estimates of $S_{1,f}$ and $J_{i,f}$

It follows from the trace theorem and the error estimate of the interpolation u_I in (2.3) that

$$\begin{aligned}|S_{1,f}|=&\Big|\int_f (u_I|_{K_1}-u_I|_{K_2})\Big\{\frac{\partial u_h}{\partial x}-\frac{\partial u}{\partial x}\Big\}dydz\Big|\\ \lesssim&\Big(h^{-\frac{1}{2}}\|u-u_I\|_{L^2(\omega_f)}+h^{\frac{1}{2}}\|\nabla_h(u-u_I)\|_{L^2(\omega_f)}\Big)\\ &\times\Big(h^{-\frac{1}{2}}\|\nabla_h(u-u_h)\|_{L^2(\omega_f)}+h^{\frac{1}{2}}\Big\|\nabla_h\Big(\frac{\partial u_h}{\partial x}-\frac{\partial u}{\partial x}\Big)\Big\|_{L^2(\omega_f)}\Big)\\ \lesssim&h^{\frac{3}{2}+s}|u|_{H^{2+s}(\omega_f)}\Big(h^{-\frac{1}{2}}\|\nabla_h(u-u_h)\|_{L^2(\omega_f)}+h^{\frac{1}{2}}\Big\|\nabla_h\Big(\frac{\partial u_h}{\partial x}-\frac{\partial u}{\partial x}\Big)\Big\|_{L^2(\omega_f)}\Big).\end{aligned} \tag{3.13}$$

Summing over the faces in $\mathcal{F}_X(\Omega)$ and recalling the error estimate of the finite element solution u_h in (2.4), this yields

$$\begin{aligned}\sum_{f\in\mathcal{F}_X(\Omega)}|S_{1,f}|\lesssim&h^{\frac{3}{2}+s}|u|_{H^{2+s}(\Omega)}\times\Big(\sum_{f\in\mathcal{F}_X(\Omega)}\Big(h^{-\frac{1}{2}}\|\nabla_h(u-u_h)\|_{L^2(\omega_f)}\\ &+h^{\frac{1}{2}}\Big\|\nabla_h\Big(\frac{\partial u_h}{\partial x}-\frac{\partial u}{\partial x}\Big)\Big\|_{L^2(\omega_f)}\Big)^2\Big)^{\frac{1}{2}}\\ \lesssim&h^{2+s}|u|^2_{H^{2+s}(\Omega)}.\end{aligned} \tag{3.14}$$

We need to bound the terms $J_{i,f}$ defined in the previous subsection. Given $f\in\mathcal{F}_X(\Omega)$ with $f=K_1\cap K_2$, we first bound the term

$$\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2}-\Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}.$$

By the interpolation space theory [4], see, for instance, [5, Chapter 12] for applications to the finite element methods, we have

$$\begin{aligned}&8h_xh_yh_z\Big|\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2}-\Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}\Big|\\ =&\Big|\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Big(\int_{x_{0,K_1}-h_x}^{x_{0,K_1}+h_x}\frac{\partial^2 u(t,p,q)}{\partial p^2}dt-\int_{x_{0,K_2}-h_x}^{x_{0,K_2}+h_x}\frac{\partial^2 u(t,p,q)}{\partial p^2}dt\Big)dpdq\Big|\\ =&\Big|\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\Big(\frac{\partial^2 u(x_{0,K_1}+t,p,q)}{\partial p^2}-\frac{\partial^2 u(x_{0,K_2}+t,p,q)}{\partial p^2}\Big)dtdpdq\Big|\\ \lesssim&8h_xh_yh_zh^{s-\frac{3}{2}}|u|_{H^{2+s}(\omega_f)}.\end{aligned} \tag{3.15}$$

Let $\Pi_f^0 v$ denote the integral average of v on f. Since

$$\int_f \Big((y-y_{0,K_1})^2-\frac{h_y^2}{3}\Big)df=0,$$

we have

$$\begin{aligned}|J_{1,f}| =&\Big|\int_f \Big(\Pi^0_{K_1}\frac{\partial^2 u}{\partial y^2}-\Pi^0_{K_2}\frac{\partial^2 u}{\partial y^2}\Big)\Big((y-y_{0,K_1})^2-\frac{h_y^2}{3}\Big)(I-\Pi_f^0)\frac{\partial u(x_m,y,z)}{\partial x}dydz\Big|\\ \lesssim& h^{2+s}|u|^2_{H^{2+s}(\omega_f)},\end{aligned}\tag{3.16}$$

which implies that

$$\sum_{f\in\mathcal{F}_X(\Omega)}|J_{1,f}|\lesssim h^{2+s}|u|^2_{H^{2+s}(\Omega)}.\tag{3.17}$$

We use (3.15) and the interpolation space theory [4,5] again to derive as

$$\begin{aligned}|J_{2,f}| =&\frac{1}{8h_xh_yh_z}\Big|\int_f\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\Big(\frac{\partial^2 u(x_{0,K_1}+t,p,q)}{\partial p^2}-\frac{\partial^2 u(x_{0,K_2}+t,p,q)}{\partial p^2}\Big)\\ &\times\Big(\frac{\partial u(x_m,y,z)}{\partial x}-\frac{\partial u(x_m,p,q)}{\partial x}\Big)dtdpdqdydz\Big|\\ \lesssim& h^s|u|^2_{H^{2+s}(\omega_f)},\end{aligned}\tag{3.18}$$

which implies that

$$\sum_{f\in\mathcal{F}_X(\Omega)}|J_{2,f}|\lesssim h^{s}|u|^2_{H^{2+s}(\Omega)}.\tag{3.19}$$

Another application of the interpolation space theory [4,5] yields

$$\begin{aligned}|J_{3,f}| =&\frac{1}{8h_xh_yh_z}\Big|\int_f\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\int_{x_{0,K_1}}^{x_{0,K_2}}\frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}\\ &\times\Big(\frac{\partial^2 u(x_m,p,q)}{\partial x\partial p}-\frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}\Big)dxdtdpdqdydz\Big|\\ \lesssim& h^s|u|^2_{H^{2+s}(\omega_f)},\end{aligned}\tag{3.20}$$

which proves that

$$\sum_{f\in\mathcal{F}_X(\Omega)}|J_{3,f}|\lesssim h^{s}|u|^2_{H^{2+s}(\Omega)}.\tag{3.21}$$

Since the terms cancel between elements and $\partial u(x_m,p,q)/\partial x$ vanishes on the boundary which is perpendicular to the y-axis, we have

$$\sum_{f\in\mathcal{F}_X(\Omega)}J_{4,f}=0.$$

Then, a summary of estimates (3.18)-(3.21) shows that

$$\sum_{f\in\mathcal{F}_X(\Omega)} S_{3,f} = \sum_{f\in\mathcal{F}_X(\Omega)} J_{1,f} + \frac{h_y^2}{3}\sum_{f\in\mathcal{F}_X(\Omega)}\sum_{\ell=2}^{5} J_{\ell,f} = \frac{h_y^2}{3}\sum_{f\in\mathcal{F}_X(\Omega)} J_{5,f} + \mathcal{O}(h^{2+s}). \tag{3.22}$$

Define

$$K_{5,f} = \frac{1}{8h_xh_yh_z}\int_f\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\int_{x_{0,K_1}}^{x_{0,K_2}}\Big(\frac{\partial^2 u(x+t,p,q)}{\partial x\partial q}\Big)^2 dxdtdpdqdydz, \tag{3.23}$$

a similar argument for the term $S_{3,f}$ can show that

$$\sum_{f\in\mathcal{F}_X(\Omega)} S_{4,f} = \frac{h_z^2}{3}\sum_{f\in\mathcal{F}_X(\Omega)} K_{5,f} + \mathcal{O}(h^{2+s}). \tag{3.24}$$

A summary of (3.6), (3.7), (3.14), (3.22), and (3.24) gives

$$\sum_{f\in\mathcal{F}_X(\Omega)}\int_f [u-u_I]\Big\{\frac{\partial u_h}{\partial \nu}\Big\}df = \frac{h_y^2}{3}\sum_{f\in\mathcal{F}_X(\Omega)} J_{5,f} + \frac{h_z^2}{3}\sum_{f\in\mathcal{F}_X(\Omega)} K_{5,f} + \mathcal{O}(h^{2+s}), \tag{3.25}$$

with $K_{5,f}$ defined as in (3.23) and $J_{5,f}$ following

$$J_{5,f} = \frac{1}{8h_xh_yh_z}\int_f\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{-h_x}^{h_x}\int_{x_{0,K_1}}^{x_{0,K_2}}\Big(\frac{\partial^2 u(x+t,p,q)}{\partial x\partial p}\Big)^2 dxdtdpdqdydz. \tag{3.26}$$

3.4 The expansion of the fourth term

We let $\mathcal{F}_{PX}(\partial\Omega)$ denote the set of boundary faces that is perpendicular to the x-axis with normal vectors $(1,0,0)$, $\mathcal{F}_{NX}(\partial\Omega)$ denote the set of boundary faces that is perpendicular to the x-axis with normal vectors $(-1,0,0)$, $\mathcal{F}_{PY}(\partial\Omega)$ the set of boundary faces that is perpendicular to the Y-axis with normal vectors $(0,1,0)$, and $\mathcal{F}_{PY}(\partial\Omega)$ the set of boundary faces that is perpendicular to the Y-axis with normal vectors $(0,-1,0)$, $\mathcal{F}_{PZ}(\partial\Omega)$ the set of boundary faces that is perpendicular to the Z-axis with normal vectors $(0,0,1)$, and $\mathcal{F}_{NZ}(\partial\Omega)$ the set of boundary faces that is perpendicular to the Z-axis with normal vectors $(0,0,-1)$. Whence, we have the following decomposition of the term I_4

$$\begin{aligned} I_4 = &\sum_{f\in\mathcal{F}_{NX}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df + \sum_{f\in\mathcal{F}_{PX}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df \\ &+\sum_{f\in\mathcal{F}_{PY}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df + \sum_{f\in\mathcal{F}_{NY}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df \\ &+\sum_{f\in\mathcal{F}_{PZ}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df + \sum_{f\in\mathcal{F}_{NZ}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df. \end{aligned} \tag{3.27}$$

We only need to consider the boundary face $f\in\mathcal{F}_{NX}(\partial\Omega)$ since the expansions for others can be coped with in a similar way. Let K be the unique element that takes f as one of its faces with the center (x_0,y_0,z_0). We use the fact that only the last two terms in (2.1) are possibly nonzero on the boundary face $f\in\mathcal{F}_{NX}(\partial\Omega)$ to get

$$\begin{aligned}\int_f (u-u_I)\frac{\partial u_h}{\partial \nu}df =&-\frac{1}{2}\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Pi_K^0\frac{\partial^2 u}{\partial y^2}((y-y_0)^2-h_y^2)\frac{\partial u_h}{\partial x}dydz\\ &-\frac{1}{2}\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Pi_K^0\frac{\partial^2 u}{\partial z^2}((z-z_0)^2-h_z^2)\frac{\partial u_h}{\partial x}dydz\\ =&S_{1,\partial\Omega,f}+S_{2,\partial\Omega,f}. \end{aligned}\tag{3.28}$$

Since the analysis for both terms $S_{1,\partial\Omega,f}$ and $S_{2,\partial\Omega,f}$ is similar, we only need to investigate the term $S_{1,\partial\Omega,f}$.

$$\begin{aligned}S_{1,\partial\Omega,f}=&-\frac{1}{2}\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Pi_K^0\frac{\partial^2 u}{\partial y^2}((y-y_0)^2-h_y^2)\Big(\frac{\partial u_h}{\partial x}-\frac{\partial u}{\partial x}\Big)dydz\\ &-\frac{1}{2}\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Pi_K^0\frac{\partial^2 u}{\partial y^2}((y-y_0)^2-h_y^2)\frac{\partial u(x_{\partial\Omega},y,z)}{\partial x}dydz\\ =&J_{1,\partial\Omega,f}+S_{3,\partial\Omega,f}, \end{aligned}\tag{3.29}$$

with $(x_{\partial\Omega},y,z)\in f$. Plus and minus the average $-2h_y^2/3$ of $((y-y_0)^2-h_y^2)$ over the face f, we have

$$\begin{aligned}S_{3,\partial\Omega,f}=&-\frac{1}{2}\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Pi_K^0\frac{\partial^2 u}{\partial y^2}\Big((y-y_0)^2-\frac{h_y^2}{3}\Big)\frac{\partial u(x_{\partial\Omega},y,z)}{\partial x}dydz\\ &+\frac{h_y^2}{3}\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\Pi_K^0\frac{\partial^2 u}{\partial y^2}\frac{\partial u(x_{\partial\Omega},y,z)}{\partial x}dydz\\ =&J_{2,\partial\Omega,f}+S_{4,\partial\Omega,f}. \end{aligned}\tag{3.30}$$

By using the fact

$$\frac{\partial u(x_{\partial\Omega},p,q)}{\partial p}\Big|_f=0,\quad \text{with }\ (x_{\partial\Omega},p,q)\in f\subset\mathcal{F}_{NX}(\partial\Omega),$$

one can follow the lines for the term $S_{5,f}$ defined as in (3.9) to analyze the term $S_{4,\partial\Omega,f}$. In particular, define

$$J_{5,\partial\Omega,f}=\frac{1}{8h_xh_yh_z}\int_f\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{x_0-h_x}^{x_0+h_x}\int_{x_{\partial\Omega}}^{t}\Big(\frac{\partial^2 u(x,p,q)}{\partial x\partial p}\Big)^2dxdtdpdqdydz,\tag{3.31}$$

one can show that

$$\sum_{f\in\mathcal{F}_{NX}(\partial\Omega)}S_{1,\partial\Omega,f}=\frac{h_y^2}{3}\sum_{f\in\mathcal{F}_{NX}}J_{5,\partial\Omega,f}+\mathcal{O}(h^{2+s}).\tag{3.32}$$

Similarly, we have

$$\sum_{f\in\mathcal{F}_{NX}(\partial\Omega)} S_{2,\partial\Omega,f} = \frac{h_z^2}{3}\sum_{f\in\mathcal{F}_{NX}} K_{5,\partial\Omega,f} + \mathcal{O}(h^{2+s}), \tag{3.33}$$

with

$$K_{5,\partial\Omega,f} = \frac{1}{8h_xh_yh_z}\int_f\int_{z_0-h_z}^{z_0+h_z}\int_{y_0-h_y}^{y_0+h_y}\int_{x_0-h_x}^{x_0+h_x}\int_{x_{\partial\Omega}}^{t}\Big(\frac{\partial^2 u(x,p,q)}{\partial x\partial q}\Big)^2 dxdtdpdqdydz. \tag{3.34}$$

Therefore, we get the expansion for the first term on the right-hand side of (3.27):

$$\sum_{f\in\mathcal{F}_{NX}(\partial\Omega)}\int_f (u-u_I)\frac{\partial u_h}{\partial\nu}df = \frac{h_y^2}{3}\sum_{f\in\mathcal{F}_{NX}} J_{5,\partial\Omega,f} + \frac{h_z^2}{3}\sum_{f\in\mathcal{F}_{NX}} K_{5,\partial\Omega,f} + \mathcal{O}(h^{2+s}). \tag{3.35}$$

3.5 The summary

In this subsection, we summarize the expansion in the previous subsections to give the main result of this section.

Lemma 3.1. *Let u and u_h be the eigenfunctions of the eigenvalue problem* (1.1) *and the discrete eigenvalue problem* (1.3), *respectively. Assume $u\in H^{2+s}(\Omega)$ with $0<s\le 1$. Then*

$$2\big(\nabla_h(u-u_I),\nabla_h u_h\big)_{L^2(\Omega)} = T_P + \mathcal{O}(h^{2+s}), \quad \textit{with}\ \ T_P>0. \tag{3.36}$$

Proof. First, one can follow the lines presented in Subsections 3.2 and 3.3 for the first term on the right-hand side of (3.5) to derive analogues of (3.25) for the other two terms on the right-hand side of (3.5). Second, one can use a similar argument in Subsection 3.4 for the term $\sum_{f\in\mathcal{F}_{NX}(\partial\Omega)}\int_f(u-u_I)(\partial u_h/\partial\nu)df$ to get analogues of (3.35) for other five terms on the right-hand side of (3.27). Finally, we insert (3.2), (3.4), (3.5), (3.25), (3.27), and (3.35) into the expansion (3.1) to prove the desired result. ☐

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