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Homotopy Perturbation Method for Time-Fractional Shock Wave Equation

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Abstract. A scheme is developed to study numerical solution of the time-fractional shock wave equation and wave equation under initial conditions by the homotopy perturbation method (HPM). The fractional derivatives are taken in the Caputo sense. The solutions are given in the form of series with easily computable terms. Numerical results are illustrated through the graph.

AMS subject classifications: 76L99, 76K99, 35L10

Key words: Partial differential equation, fractional derivative, shock wave equation, homotopy perturbation method.

1 Introduction

In recent years, considerable interest has been devoted to the study of the fractional calculus during the past decades and their numerous applications in the area of physics and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science, probability and statistics, electrochemistry of corrosion, chemical physics, and signal processing are well described by differential equations of fractional order [1–3]. The HPM is the new method for finding the approximate analytical solution of linear and nonlinear problems [4, 5] and successfully applied to solve nonlinear wave equation. The fractional diffusion equation with absorbent term and external force through HPM is analyzed in [6]. The proof of the existence of the attractor for the one-dimensional viscous Fornberg-Whitham equation is studied by [7]. The solution of shock wave equation is examined by ADM and HPM in [8, 9]. In 2010, Golbabai and Sayevand [10] applied the HPM to solve the

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multi-order time fractional differential equations and space-time fractional solidification in a finite slab solved by Singh et al. [11]. Recently, Gupta and Singh [12] used the HPM to solve the time-fractional Fornberg-Whitham equation. Recently, many new approaches for finding the exact solutions to nonlinear equations have been proposed, for example, Exp-function method [13], homotopy analysis method [14], and reduced differential transform method [15] and so on. All methods, mentioned above, have limitations in their applications.

In present article, we implement the Homotopy perturbation method for obtaining analytical and numerical solutions of the shock wave equation with time-fractional derivatives. This equation can be written in operator form as [16–18]

$$u_t^\alpha(x,t) = \Big(\frac{1}{c_0} - \frac{\gamma+1}{2}\frac{u}{c_0^2}\Big)u_x = 0, \quad t>0, \;\; x\in R, \;\; 0<\alpha\le 1, \tag{1.1}$$

with initial condition

$$u_0(x,0) = \exp\Big(-\frac{x^2}{2}\Big), \tag{1.2}$$

where c_0 is constant and γ is specific heat.

In [8,9], it is shown that if $c_0 \ge (\gamma+1)u/2$ then a series solution can be obtained and it is given by

$$u(x,t) = \sum_0^\infty \frac{(n+1)^{\frac{n}{2}}}{(n+1)!} H_n\big(\sqrt{n+1}\big)\exp\Big[-\frac{1}{2}\Big(x-\frac{t}{2}\Big)^2(n+1)\Big], \tag{1.3}$$

where $B=(\gamma+1)/2c_0^2$ and $H_n(\cdot)$ is the Hermit polynomial of order n.

2 Preliminaries and notations

In this section, we have given some definitions and properties of the fractional calculus [1] which are used further in this paper.

Definition 2.1. *A real function $f(t)$, $t>0$ is said to be in the space C_μ, $\mu\in\mathfrak{R}$, if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu$, $n\in N$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator (J_t^α) of order $\alpha\ge 0$, of a function $f\in C_\mu$, $\mu\ge -1$ is defined as* [2]

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} f(\xi)d\xi, \quad \alpha>0, \;\; t>0,$$
$$J_t^0 f(t) = f(t),$$

where $\Gamma(\alpha)$ is the well-known gamma function. Some of the properties of the operator J_t^α, which we will need here, are as follows: for $f\in C_\mu$, $\mu\ge -1$, $\alpha,\beta\ge 0$ and $\gamma\ge -1$,