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Closed-Form Expression for the Exact Period of a Nonlinear Oscillator Typified by a Mass Attached to a Stretched Wire

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Abstract. The exact analytical expression of the period of a conservative nonlinear oscillator with a non-polynomial potential, is obtained. Such an oscillatory system corresponds to the transverse vibration of a particle attached to the center of a stretched elastic wire. The result is given in terms of elliptic functions and validates the approximate formulae derived from various approximation procedures as the harmonic balance method and the rational harmonic balance method usually implemented for solving such a nonlinear problem.

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Key words: Nonlinear oscillator, exact period, elliptic integral.

1 Introduction

We consider a one-dimensional oscillator of motion equation,

$$\ddot{x} + x - \frac{\lambda x}{\sqrt{x^2+1}} = 0, \qquad 0 \le \lambda \le 1. \tag{1.1}$$

This nonlinear ODE describes in a dimensionless form, for instance, the transverse vibration of a particle attached to the centre of a stretched elastic wire. Such an oscillator has been first analyzed by Mickens [1], and more recently reconsidered, among many others, by Beléndez et al. [2], Sun et al. [3] and Gimeno et al. [4]. The parameter λ is a geometrical deformation characteristic of the wire when the oscillator is at rest. For such a conservative system, the total energy

$$E = \frac{\dot{x}^2}{2} + \frac{x^2}{2} - \lambda\sqrt{x^2+1},$$

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is conserved and the motion of the particle is oscillatory for any energy value $E > -\lambda$. Notice that the energy is, in principle, defined up to a constant: for simplifying further the calculations, our convention is to fix the energy minimum at the value $-\lambda$, and not zero as customarily assumed in the works previously indicated.

It is known that the oscillation period depends on energy and is given by the integral [5],

$$T_\lambda(E) = \sqrt{2}\int_{-x_m(E)}^{x_m(E)} \Big(E - \frac{1}{2}x^2 + \lambda\sqrt{x^2+1}\Big)^{-\frac{1}{2}} dx, \tag{1.2}$$

where the turning points $-x_m$ and x_m $(x_m > 0)$ of the trajectory at energy E are the two real solutions of the equation,

$$E - \frac{1}{2}x^2 + \lambda\sqrt{x^2+1} = 0,$$

such that,

$$x_m = \sqrt{2}\sqrt{\sqrt{\lambda^2(\lambda^2+2E+1)} + (\lambda^2+E)}. \tag{1.3}$$

In addition, it is worth noting that if $\lambda \geq 1/2$ and $-\lambda < E \leq -1/2$, the latter equation has also the two following imaginary solutions,

$$\pm i\kappa = \pm i\sqrt{2}\sqrt{\sqrt{\lambda^2(\lambda^2+2E+1)} - (\lambda^2+E)}, \tag{1.4}$$

such that $0 < \kappa \leq 1$ (equality when $E = -1/2$).

The calculation of Eq. (1.2) is not straightforward, and to the best of our knowledge, such an exact result cannot be found anywhere. Only approximate solutions exist which are derived, for instance, by means of the harmonic balance method [2,3], the rational balance method [4], the energy balance method [6], the parameter-expansion method [7] or using a variational approach [8] when applied for solving Eq. (1.1).

In this note, we derive an exact closed-form expression for the period function $T_\lambda(E)$ in terms of complete elliptic integrals of the first and third kinds. Such a result is obtained thanks to appropriate integrations in the complex plane, as detailed in Sections 2 and 3. We summarize our main result in Section 4 and give, finally in the last section, accurate estimates of the period for small and large amplitude oscillations which validate the nonlinear analytical methods usually employed for solving this problem.

2 Integration in the complex plane

2.1 case I : for $0 < \lambda < \frac{1}{2}$

The function of a complex variable $z = x + iy$,

$$f(z) = \Big(E - \frac{1}{2}z^2 + \lambda\sqrt{z^2+1}\Big)^{-\frac{1}{2}}, \tag{2.1}$$

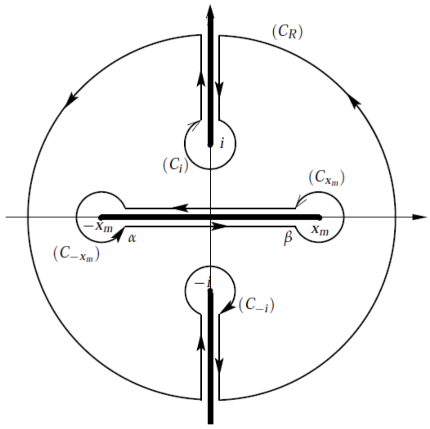

Figure 1: Integration contour (γ_I); the branch lines are depicted in thick lines.

has branch points at $z = \pm x_m, \pm i$, and no other singularities including the point at infinity. We may therefore introduce the two branch lines : (i) from $z = -x_m$ to $z = x_m$, and (ii) from $z = \pm i$ to infinity, and consider the integration contour (γ_I) drawn in Fig. 1, in such a manner that $f(z)$ has a real positive determination on the segment (α, β). $(C_{\pm i})$ and $(C_{\pm x_m})$ are circles of radius ϵ infinitely small, and (C_R) a circle of radius R infinitely large. First, we can readily show that the integrals round the four small circles tend to zero as $\epsilon \to 0$, while as $R \to +\infty$,

$$\oint_{C_R} f dz \longrightarrow 2\pi\sqrt{2}.$$

Secondly, the integral along the two horizontal lines gives, as $\epsilon \to 0$,

$$\int_{-x_m}^{x_m} \Big(E - \frac{1}{2}x^2 + \lambda\sqrt{x^2+1}\Big)^{-\frac{1}{2}} dx + \int_{x_m}^{-x_m} -\Big(E - \frac{1}{2}x^2 + \lambda\sqrt{x^2+1}\Big)^{-\frac{1}{2}} dx = \sqrt{2}T_\lambda(E).$$

Note that the outer square root has changed of determination due to the rotation round each branch point $z = \pm x_m$ (which is not the case for the inner square root) and is negative real-valued on the upper segment.

Thirdly, the integral along the two upper vertical lines gives, considering this time the change of determination of the inner square root on both sides of the branch cut $(i, i\infty)$,

$$\begin{aligned}&\int_1^{+\infty} \Big(E + \frac{1}{2}y^2 + i\lambda\sqrt{y^2-1}\Big)^{-\frac{1}{2}} i dy + \int_1^{+\infty} -\Big(E + \frac{1}{2}y^2 - i\lambda\sqrt{y^2-1}\Big)^{-\frac{1}{2}} i dy \\ =&2\,\mathrm{Im}\Big(\int_1^{+\infty} \Big(E + \frac{1}{2}y^2 - i\lambda\sqrt{y^2-1}\Big)^{-\frac{1}{2}} dy\Big).\end{aligned}$$

The result is the same for the integral along the two lower vertical lines.

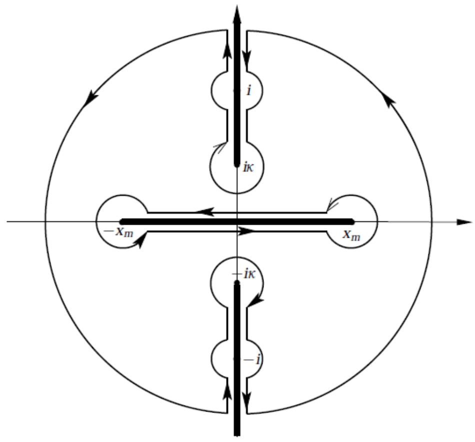

Figure 2: Integration contour (γ_{II}).

Finally, by the Cauchy's theorem, we may conclude that,

$$
\begin{aligned}
T_\lambda(E) =& 2\pi + 2\sqrt{2}\,\mathrm{Im}\Big(\int_1^{+\infty}\Big(E+\frac{1}{2}y^2 - i\lambda\sqrt{y^2-1}\Big)^{-\frac{1}{2}}dy\Big)\\
=& 2\pi + \ \mathrm{Im}\Big(\int_0^{+\infty} 4y\Big(\sqrt{1+y^2}\sqrt{(y-i\lambda)^2+\lambda^2+2E+1}\Big)^{-1}dy\Big). \qquad (2.2)
\end{aligned}
$$

2.2 case II : for $\lambda \geq \frac{1}{2}$

- if $E \geq -\frac{1}{2}$, the function Eq. (2.1) has the same singularities as preceding, and the result Eq. (2.2) remains formally valid.
- if $E < -\frac{1}{2}$, the function Eq. (2.1) has two supplementary branch points $z = \pm i\kappa$ given by Eq. (1.4). Consequently, in order to make $f(z)$ uniform, we must now introduce the two branch lines: (i) from $z = -x_m$ to $z = x_m$, (ii) from $z = \pm i\kappa$ to infinity, and consider in this case the integration contour (γ_{II}) drawn in Fig. 2, by choosing the same determination convention than previously.

The calculation of all the path integrals gives the same results as previously: in particular, the integrals round the small circles and semi-circles tend to zero as $\epsilon \to 0$, and the integral along the four vertical segments lying between $(i\kappa, i)$ and $(-i, -i\kappa)$ writes after simplification,

$$
4i\int_\kappa^1 \Big(E+\frac{1}{2}y^2+\lambda\sqrt{1-y^2}\Big)^{-\frac{1}{2}}dy. \qquad (2.3)
$$

Note that this latter integral is real-valued since

$$
E+\frac{1}{2}y^2+\lambda\sqrt{1-y^2} < 0, \quad \text{for } \kappa < y < 1.
$$

Therefore, we obtain,

$$\begin{aligned}T_\lambda(E) =&2\pi + 2\sqrt{2}\,\mathrm{Im}\Big(\int_1^{+\infty}\Big(E+\frac{1}{2}y^2-i\lambda\sqrt{y^2-1}\Big)^{-\frac{1}{2}}dy\Big)\\&+2\sqrt{2}\int_\kappa^1\Big(-E-\frac{1}{2}y^2-\lambda\sqrt{1-y^2}\Big)^{-\frac{1}{2}}dy\\=&2\pi+\ \mathrm{Im}\Big(\int_0^{+\infty}4y\Big(\sqrt{1+y^2}\sqrt{(y-i\lambda)^2+2E+1+\lambda^2}\Big)^{-1}dy\Big)\\&+\int_0^{\sqrt{1-\kappa^2}}4y\Big(\sqrt{1-y^2}\sqrt{(y-\lambda)^2-(2E+1+\lambda^2)}\Big)^{-1}dy.\end{aligned}\tag{2.4}$$

At this step, our objective is not completely achieved, but the obtained results Eqs. (2.2) and (2.4) clearly show the possibility of formulating the period function by using solely elliptic integrals. That is done in the next section.

3 Results in terms of elliptic integrals

Let us define the following real parameters,

$$c=\sqrt{1-\kappa^2}=\lambda-\sqrt{2E+1+\lambda^2},\qquad a=\lambda+\sqrt{2E+1+\lambda^2}.$$

Since $E>-\lambda$ and $0\le\lambda\le 1$, one may check that in any cases, $c<1<a$.

3.1 Step I

In the case II, for $E<-1/2$, one has,

$$0<c<1<a.$$

Hence, the real-valued integral Eq. (2.3) may be expressed as,

$$\mathcal{I}_1=\int_0^c\frac{4y}{\sqrt{(a-y)(1-y)(c-y)(y+1)}}dy.$$

Therefore [9, p. 273],

$$\begin{aligned}\mathcal{I}_1=&\frac{4\sqrt{2}}{\sqrt{a-c}}\Big\{(c-1)\Pi\Big(\sin^{-1}\Big(\sqrt{\frac{2c}{c+1}}\Big),\frac{c+1}{2},\sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big)\\&+F\Big(\sin^{-1}\Big(\sqrt{\frac{2c}{c+1}}\Big),\sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big)\Big\},\end{aligned}$$

Π and F denoting elliptic integrals of the third and first kinds respectively, defined in the Gradshteyn et al.'s conventions [9], by

$$\begin{aligned}\Pi(\varphi,n,k)&=\int_0^{\sin\varphi}\frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},\\F(\varphi,k)&=\int_0^{\sin\varphi}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.\end{aligned}$$

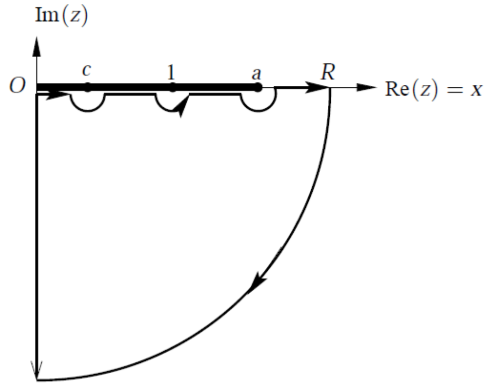

Figure 3: Integration contour for the case (α); the point c is dropped for the case (β).

3.2 Step II

In both cases I and II, the integral in Eq. (2.2) may be written as,

$$\mathrm{Im}(\mathcal{I}_2) = \mathrm{Im}\Big(\int_0^{+\infty} \frac{4y}{\sqrt{(y-i)(y+i)}\sqrt{(y-ia)(y-ic)}}dy\Big),$$

and by the change of variable $y = iz$, becomes a line integral along the semi-imaginary axis Δ from 0 to $-i\infty$,

$$\mathrm{Im}(\mathcal{I}_2) = \mathrm{Im}\Big(\int_\Delta \frac{-4z}{\sqrt{(z+1)(1-z)}\sqrt{(z-c)(a-z)}}dz\Big).$$

By the Cauchy's theorem applied to the contour displayed in Fig. 3 (the small semi-circles are of radius infinitely small and the big one of radius R infinitely large), we formally obtain,

$$\mathrm{Im}(\mathcal{I}_2) = -2\pi + \mathrm{Im}\Big(\int_0^{+\infty} \frac{-4x}{\sqrt{(x+1)(1-x)}\sqrt{(x-c)(a-x)}}dx\Big),$$

where $\int_0^{+\infty}$ means, depending on whether $c > 0$ or not,

$$(\alpha): \int_0^{+\infty} = \int_0^c + \int_c^1 + \int_1^a + \int_a^{+\infty}, \qquad \text{if } \lambda > \frac{1}{2} \text{ and } E < -\frac{1}{2} \text{ (case II)},$$

$$(\beta): \int_0^{+\infty} = \int_0^1 + \int_1^a + \int_a^{+\infty}, \qquad \text{if not.}$$

All these integrals can be evaluated in terms of elliptic integrals [9]. Note in passing that in the latter case, the integrals $\int_0^1$ and $\int_a^{+\infty}$ are real-valued, and may be dropped. After simplification, we thus obtain,

$$(\alpha) \quad \mathrm{Im}(\mathcal{I}_2) = -2\pi - \mathcal{I}_1 + \frac{4\sqrt{2}}{\sqrt{a-c}}\Big\{(a+1)\Pi\Big(\frac{1-a}{2}, \sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big) - K\Big(\sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big)\Big\},$$

$$(\beta) \quad \mathrm{Im}(\mathcal{I}_2) = -2\pi + \frac{4\sqrt{2}}{\sqrt{a-c}}\Big\{(a+1)\Pi\Big(\frac{1-a}{2}, \sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big) - K\Big(\sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big)\Big\}.$$

As usual,

$$K(k) = F\Big(\frac{\pi}{2}, k\Big) \quad \text{and} \quad \Pi(n,k) = \Pi\Big(\frac{\pi}{2}, n, k\Big),$$

denote complete elliptic integrals of the first and third kinds respectively. In particular, if $E = \lambda$ (thus, $c = -1$), one has

$$\text{Im}(\mathcal{I}_2) = 2\pi\Big(1 - \frac{\sqrt{2}}{\sqrt{a+1}}\Big).$$

4 Final exact formula

For $0 < \lambda \leq 1$ and any $E > -\lambda$, the main result is:

$$T_\lambda(E) = \frac{4\sqrt{2}}{\sqrt{a-c}}\Big\{(a+1)\Pi\Big(\frac{1-a}{2}, \sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big) - K\Big(\sqrt{\frac{(a-1)(c+1)}{2(a-c)}}\Big)\Big\}. \tag{4.1}$$

This exact period is plotted in Fig. 4 for different values of λ.

In particular, if $\lambda > 1/2$,

$$T_\lambda\Big(-\frac{1}{2}\Big) = \frac{4}{\sqrt{\lambda}}\Big\{(2\lambda+1)\Pi\Big(\frac{1-2\lambda}{2}, \sqrt{\frac{2\lambda-1}{4\lambda}}\Big) - K\Big(\sqrt{\frac{2\lambda-1}{4\lambda}}\Big)\Big\},$$

and for any $0 < \lambda \leq 1$,

$$T_\lambda(\lambda) = 2\pi\Big(2 - \frac{1}{\sqrt{\lambda+1}}\Big). \tag{4.2}$$

If $\lambda = 0$, the oscillator is harmonic and isochronous: the direct calculation of the period from Eq. (1.1), or by taking the limit of Eq. (4.1) when the parameter λ tends to zero, gives as expected, $T_\lambda(E) = 2\pi$ for any $E > 0$. However, we shall prove in the next section that

$$\lim_{E\to+\infty} T_\lambda(E) = 2\pi, \quad \forall\lambda,$$

$$\lim_{E\to-\lambda} T_\lambda(E) = \begin{cases} \dfrac{2\pi}{\sqrt{1-\lambda}}, & \text{if } \lambda \neq 1, \\ +\infty, & \text{if } \lambda = 1. \end{cases}$$

As an application, we give in Appendix (Table 1), for different values of λ and E, the exact value of the period T_λ compared to the approximate ones T_a as computed in [4] which constitute, to date, the best accurate results obtained by the known analytical approximation methods.

5 Asymptotic expressions

A rapid inspection of the motion equation Eq. (1.1) shows that the oscillator is quasi-harmonic for,

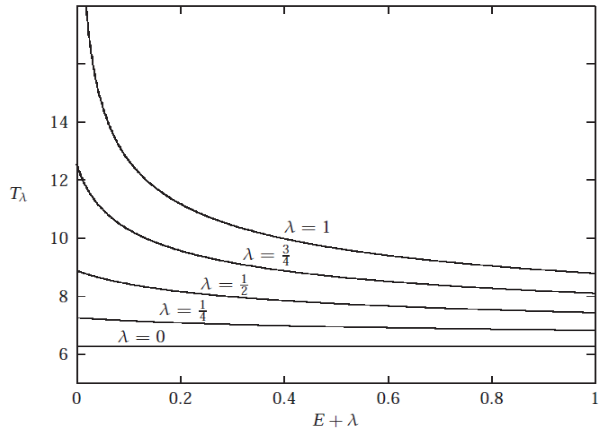

Figure 4: Graph of the exact period T_λ in function of $E+\lambda$, with $\lambda = 0,\ 1/4,\ 1/2,\ 3/4,\ 1$.

- large amplitude (or large energy) $\forall\lambda$:

$$\ddot{x} + x \underset{E\to+\infty}{\simeq} \pm\lambda \quad \text{and} \quad T_\lambda(E) \sim 2\pi.$$

- small amplitude around the origin (or $E+\lambda \ll 1$) if $\lambda \neq 1$:

$$\ddot{x} + (1-\lambda)x \underset{E\to-\lambda}{\simeq} 0 \quad \text{and} \quad T_\lambda(E) \sim \frac{2\pi}{\sqrt{1-\lambda}}.$$

If $\lambda = 1$, the oscillator is exceptionally "truly nonlinear" for small amplitude,

$$\ddot{x} + \frac{1}{2}x^3 \simeq 0 \quad \text{and} \quad T_1(E) \underset{E\to-1}{\longrightarrow} +\infty.$$

All these approximations agree with the results of the previous section. But, it is necessary to precise these asymptotic properties by giving proper estimates of the period for each case.

5.1 Large amplitude

For $E \gg 1$, the easiest way to obtain an asymptotic expression for $T_\lambda(E)$ is to start from the integral Eq. (1.2) using the change of variable

$$x = x_m X,$$

x_m given by Eq. (1.3),

$$T_\lambda(E) = 2\sqrt{2}\int_0^1 x_m\Big(E - \frac{1}{2}x_m^2X^2 + \lambda\sqrt{x_m^2X^2+1}\Big)^{-\frac{1}{2}} dX. \tag{5.1}$$

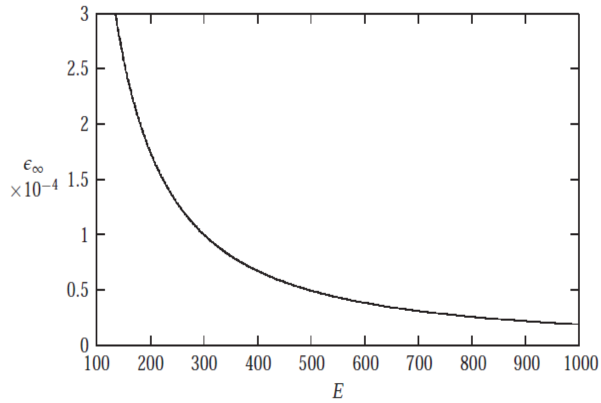

Figure 5: Graph of the error ϵ_∞ in function of energy for $\lambda = 1$.

A series expansion in powers of $1/E$ readily gives

$$\begin{aligned} T_\lambda(E) \sim T_\lambda^\infty(E) =& 2\sqrt{2}\int_0^1 dX\Big\{\frac{\sqrt{2}}{\sqrt{1-X^2}} + \lambda\sqrt{\frac{1}{E}}\frac{1}{(X+1)\sqrt{1-X^2}} \\ &+ \frac{\lambda^2}{2\sqrt{2}}\frac{1}{E}\frac{1-2X}{(X+1)^2\sqrt{1-X^2}} + \mathcal{O}\Big(\frac{1}{E^{3/2}}\Big)\Big\}. \end{aligned}$$

Thus, an estimate of Eq. (5.1) for large energy, in agreement with the result found in [2], here reads,

$$T_\lambda^\infty(E) = 2\pi\Big(1 + \frac{\lambda}{\pi}\sqrt{\frac{2}{E}} + \mathcal{O}\Big(\frac{1}{E^{3/2}}\Big)\Big). \tag{5.2}$$

Fig. 5 shows the excellent accuracy of this estimate in the most disadvantageous case viz. when $\lambda = 1$: indeed, the relative error,

$$\epsilon_\infty = \Big|\frac{T_1^\infty(E) - T_1(E)}{T_1(E)}\Big|,$$

is less than 10^{-4} for $E > 300$ ($T_1(E)$ is given by Eq. (4.1)).

5.2 Small amplitude for $\lambda \neq 1$

On the other hand, for $E + \lambda \ll 1$, one has $x_m \ll 1$ and Eq. (5.1) may be expressed as,

$$\begin{aligned} T_\lambda(E) =& 2\sqrt{2}\int_0^1 x_m\Big[E - \frac{1}{2}x_m^2X^2 + \lambda\Big(1 + \frac{1}{2}x_m^2X^2 - \frac{1}{8}x_m^4X^4 + \cdots\Big)\Big]^{-\frac{1}{2}} dX \\ =& 2\sqrt{2}\int_0^1 \Big(\frac{E+\lambda}{x_m^2} - \frac{1-\lambda}{2}X^2 - \frac{\lambda}{8}x_m^2X^4 + \cdots\Big)^{-\frac{1}{2}} dX. \end{aligned} \tag{5.3}$$

Since,

$$\begin{cases} \dfrac{E+\lambda}{x_m^2} = \dfrac{1-\lambda}{2} + \dfrac{\lambda}{4(1-\lambda)}(E+\lambda) + \mathcal{O}(E+\lambda)^2, \\ x_m^2 = \dfrac{2}{1-\lambda}(E+\lambda) + \mathcal{O}(E+\lambda)^2, \end{cases}$$

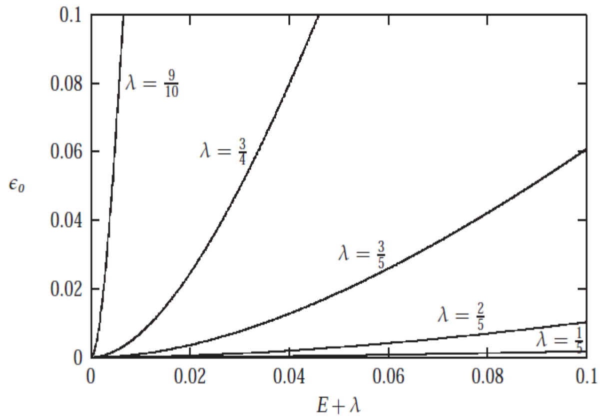

Figure 6: Graph of the error ϵ_o in function of $E+1$ for $\lambda = 1/5,\ 2/5,\ 3/5,\ 3/4,\ 9/10$.

a first order expansion of the period in powers of $(E+\lambda)$ may be obtained as follows,

$$T_\lambda(E) \sim T_\lambda^o(E) = \frac{4}{\sqrt{1-\lambda}} \int_0^1 dX \Big\{ \frac{1}{\sqrt{1-X^2}} - \frac{\lambda}{4(1-\lambda)^2} \frac{1+X^2}{\sqrt{1-X^2}} (E+\lambda) + \mathcal{O}(E+\lambda)^2 \Big\}.$$

Thus, an estimate of the small oscillation period around the equilibrium is $(0 \le \lambda < 1)$,

$$T_\lambda^o(E) = \frac{2\pi}{\sqrt{1-\lambda}} - \frac{3\pi}{4} \frac{\lambda}{(1-\lambda)^{5/2}} (E+\lambda) + \mathcal{O}(E+\lambda)^2. \tag{5.4}$$

Fig. 6 displays the relative error,

$$\epsilon_o = \Big| \frac{T_\lambda^o(E) - T_\lambda(E)}{T_\lambda(E)} \Big|, \tag{5.5}$$

for $E+\lambda \le 0.1$ ($T_\lambda(E)$ is given by Eq. (4.1)), and clearly shows that the latter estimate has concrete sense only for small values of λ.

5.3 Small amplitude for $\lambda = 1$

In that case,

$$\begin{cases} \dfrac{E+1}{x_m^2} = \dfrac{\sqrt{2}}{4}\sqrt{E+1} - \dfrac{1}{4}(E+1) + O(E+1)^{\frac{3}{2}} \\ x_m^2 = 2\sqrt{2}\sqrt{E+1} + 2(E+1) + O(E+1)^{\frac{7}{2}} \end{cases},$$

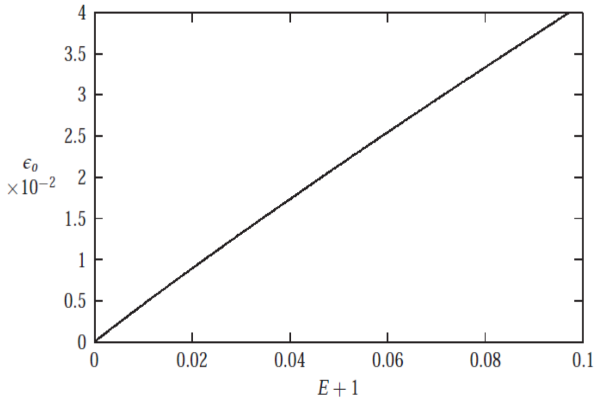

Figure 7: Graph of the error ϵ_0 in function of $E+1$ for $\lambda = 1$.

so Eq. (5.3) may be recasted as,

$$\begin{aligned} T_1(E) &= 4\left(\frac{2}{E+1}\right)^{\frac{1}{4}} \int_0^1 \frac{dX}{\sqrt{1 - X^4 - \frac{\sqrt{2}}{2}\sqrt{E+1}(1-X^2)(2X^4+X^2+1)+\dots}} \\ &= 4\left(\frac{2}{E+1}\right)^{\frac{1}{4}} \int_0^1 dX \left\{ \frac{1}{\sqrt{1-X^4}} + \frac{1}{2\sqrt{2}}\sqrt{E+1}\frac{2X^4+X^2+1}{(1+X^2)\sqrt{1-X^4}} \right. \\ &\quad \left. + \frac{3}{16}(E+1)\frac{(2X^4+X^2+1)^2}{(1+X^2)^2\sqrt{1-X^4}} + O(E+1)^{\frac{3}{2}} \right\}. \end{aligned}$$

Thus, we finally obtain the estimate (see also, [2]),

$$\begin{aligned} T_1(E) \sim T_1^o(E) &= \sqrt{\pi}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\left(\frac{2}{E+1}\right)^{\frac{1}{4}} + \frac{6\pi\sqrt{\pi\sqrt{2}}}{\Gamma^2\left(\frac{1}{4}\right)}(E+1)^{\frac{1}{4}} \\ &\quad + \frac{17\Gamma^4\left(\frac{1}{4}\right) - 192\pi^2}{16\sqrt{\pi\sqrt{2}}\Gamma^2\left(\frac{1}{4}\right)}(E+1)^{\frac{3}{4}} + O(E+1)^{\frac{5}{4}}. \end{aligned} \tag{5.6}$$

Fig. 7 shows the excellent accuracy of this result compared to the exact result Eq. (4.1) (with $\lambda = 1$): the relative error Eq. (5.5) is less than 4% for $E+1 \leq 0.1$.

6 Conclusions

Although very classical, the integration method implemented in this note is relatively little used, while it could be a convenient way for solving numerous problems. Nonetheless, for our particular example studied here, it is highly probable there exists other neater methods which would shorten and simplify even more the calculation. We leave it to the reader to improve it. The comparison of the obtained exact period with the approximate results found in the literature (notably, in [3] and [4]), shows the excellent performance (in terms of accuracy and convergence) of analytical approximation methods proposed by these authors.

Let us point out for ending, that, beyond its numerical interest discussed above, the knowledge of the exact period in function of energy- and any related physical quantity as the action integral [5],

$$S_\lambda(E) = \sqrt{2}\int_{-x_m(E)}^{x_m(E)} \sqrt{E - V(x)}dx = \frac{1}{2}\int_{-\lambda}^{E} T_\lambda(\varepsilon)d\varepsilon,$$

may turn out to be essential for studying the physical properties of our oscillator, and any applications where the energy plays a central role (semiclassical quantization of the oscillator, thermodynamic functions etc, see e.g., [10]).

Appendix

Using a modified rational harmonic balance method, Gimeno et al. have given in [4] the approximate periods T_a which are the best accurate results available from the literature at the present time. In Table 1, T_a is compared to the exact value T_λ computed from Eq. (4.1) for different values of λ and oscillation amplitude x_m (related to the energy E by Eq. (1.3)). The evaluation of the relative error defined as,

$$\epsilon(\%) = \frac{|T_a - T_\lambda|}{T_\lambda} \times 100,$$

confirms the high quality of the used analytical approximation method for solving Eq. (1.1).

Table 1: Comparison of the exact period with the approximate periods derived by Gimeno et al. [4].

λ	x_m	E	T_λ	T_a	$\epsilon(\%)$
	0.4	-0.0277033	6.603056	6.603056	0.000000
0.10	1.0	0.3585786	6.537508	6.537512	0.000061
	10.0	48.9950124	6.322939	6.322942	0.000047
	0.4	-0.4585165	8.653029	8.653036	0.000081
0.50	1.0	-0.2071068	7.992133	7.992415	0.0035
	10.0	44.9750622	6.490208	6.490294	0.0013
	0.4	-0.7277747	11.652496	11.652575	0.00068
0.75	1.0	-0.5606602	9.625405	9.627069	0.017
	10.0	42.4625933	6.602092	6.602306	0.0032
	0.4	-0.9431813	19.780733	19.782607	0.0095
0.95	1.0	-0.8435029	12.075277	12.084201	0.074
	10.0	40.4526182	6.696117	6.696489	0.0055
	0.4	-0.9970330	27.418114	27.428923	0.039
1.00	1.0	-0.9142136	13.066809	13.082014	0.12
	10.0	39.9501244	6.720292	6.720713	0.0062

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