The Reflection of Magneto-Thermoelastic P and SV Waves at a Solid Half Space Using Dual-Phase-Lag Model

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Abstract. The dual-phase-lag heat transfer model is employed to study the reflection phenomena of P and SV waves from a surface of a semi-infinite magneto-thermoelastic solid. The ratios of reflection coefficients to that of incident coefficients are obtained for P- and SV-wave cases. The results for partition of the energy for various values of the angle of incidence are computed numerically under the stress-free and rigidly fixed thermally insulated boundaries. The reflection coefficients are depending on the angle of incidence, magnetic field, phase lags and other material constants. Results show that the sum of energy ratios is unity at the interface. The results are discussed and depicted graphically.

AMS subject classifications: 73B30, 35Q99

Key words: Reflection, dual-phase-lag model, thermoelastici waves, partition of energy, magneto-thermoelasticity.

1 Introduction

Since the early 1960's there has been an increased usage of composite materials in variety of commercial, aerospace, and military structural configurations involving extreme temperature environments. Therefore, during the past three decades, wide spread attention has been given to thermoelasticity theories which admit a finite speed for the propagation of thermal signals. In contrast to the conventional theories based on parabolic-type heat equation, these theories involve a hyperbolic-type heat equation and are referred to as generalized theories. Various authors have formulated these generalized theories on different grounds. For example Lord and Shulman [6] have developed a theory based on a modified heat conduction law which involves heat

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flux rate. This thermoelastic theory includes the finite velocity of thermal wave by correcting the Fourier thermal conduction law by introducing one relaxation time of thermoelastic process. Green and Lindsay [4] formulated a more rigorous theory by including a temperature rate among the constitutive variables; they are considered the finite velocity of the thermal wave by correcting the energy equation and Duhamel-Neumann relation, by introducing two relaxation times of the thermal process. Green and Naghdi [5] developed a theory describing the behavior of a thermoelastic body. This theory is usually called "without energy dissipation".

Tzou [12,13] proposed the dual-phase-lag **(DPL)** model, which describes the interactions between phonons and electrons on the microscopic level as retarding sources causing a delayed response on the macroscopic scale. For macroscopic formulation, it would be convenient to use the DPL mode to investigate of the micro-structural effect on the behavior of heat transfer. The physical meanings and the applicability of the **DPL** model have been supported by the experimental results [14]. The dual-phase-lag **(DPL)** proposed by Chandrasekharaiah and Tzou [1,14] is such a modification of the classical thermoelastic model in which the Fourier law is replaced by an approximation to a modified Fourier law with two different time translations: a phase-lag of the heat flux τ_q and a phase-lag of the temperature gradient τ_θ. A Taylor series approximation of the modified Fourier law, together with the remaining field equations lead to a complete system of equations describing a dual-phase-lag thermoelastic model. The model transmits thermoelastic disturbance in a wave- like manner if the approximation is linear with respect to τ_q and τ_θ, and $0 \leq \tau_\theta < \tau_q$; or quadratic in τ_q and linear in τ_θ, with $\tau_q > 0$ and $\tau_\theta > 0$. This theory is developed in a rational way to produce a fully consistent theory which is able to incorporate thermal pulse transmission in a very logical manner.

In the last few decades a new domain has been developed which investigates the interactions between strain and electromagnetic fields. This discipline is called magneto-elasticity. A stimulus for its development was the possibility of its applications to geophysical problems, certain topics in optics and acoustics, investigations on damping of acoustic waves in a magnetic field, etc. The theory of magneto-thermoelasticity is concerned with the influence of the magnetic field on the elastic and thermoelastic deformations of a solid body. This theory has aroused much interest in recent years, because of its application in various branches of science and technology. With the rapid development of polymer science and plastic industry, as well as the wide use of materials under high temperature in modern technology and application of biology and geology in engineering, the theoretical study and applications in viscoelastic materials has become an important task for solid mechanics.

The subject of wave propagation and their reflection and transmission from interfaces in an elastic medium is of great interest since long. These studies help us manifold e.g., the elastic waves propagating through the Earth (called seismic waves) have to travel through different layers and interfaces. The velocities of these waves are influenced by the properties of the layer through which they travel and whenever these waves come across the discontinuities between different layers, the phenomena

of reflection and transmission take place. The signals of these reflected and transmitted waves are not only helpful in providing information about the internal structures of the Earth but also helpful in exploration of the valuable materials such as minerals, crystals and metals etc. The technique of waves propagation is one of the most suitable in terms of time saving and economy. So the problems of elastic wave propagation and their reflection and transmission are of great help to geophysics, engineers in mineral companies and future researchers in the pertinent area. The studies of reflection of waves are of great interest to seismologists. Such studies help them to obtain knowledge about the rock structures as well as their elastic properties and at the same time information regarding minerals and fluids present inside the earth.

The interaction of electromagnetic fields with the motion of a deformable solid is attracting greater attention by many investigators. Among many important problems considered in such studies, the problems of elastic wave propagation in the presence of a steady magnetic field were investigated by Othman and Song [7]. The reflection of magneto-thermoelastic waves under generalized thermoelasticity theory was carried out to study the reflection of plane harmonic waves from a semi-infinite elastic solid in a vacuum. Sinha and Elsibai [8, 9] studied the effect of the two relaxation times on the reflection of thermoelastic waves at a homogeneous, isotropic and thermally conducting elastic solid half space in the context of generalized thermoelasticity with two relaxation times. Singh and Khurana [11] investigated correctly the problem of reflection of P- and SV-waves at a free surface of a monoclinic elastic half space. Singh and Khurana [10] also investigated the reflection and transmission of P- and SV-waves at the interface between two monoclinic elastic half-spaces. Chaudhary et al. [2, 3] studied the problems of transmission of a plane SH-wave through a self-reinforced elastic slab sandwiched between two elastic solid half-spaces.

In the present paper, we intend to study of magneto-thermoelastic waves in a homogeneous isotropic, conducting semi-infinite elastic solid nearby a vacuum in the context of **DPL** theory of thermoelasticity. The expressions for the reflection coefficients, which are the ratios of the amplitudes of the reflected waves to the amplitude of the incident wave and the partition of energy are obtained in the case of stress free surface boundary condition. The analytical expressions for amplitudes of displacement, perturbed magnetic field and temperature change have been also derived. The effect of various parameters such as magnetic field, coupling parameter and phase-lags and the angle at which the wave crosses the magnetic field, on these coefficients. The results obtained theoretically have been computed numerically and presented graphically. Some earlier results of other researchers have been reduced as a particular case of the present formulation.

2   Formulation of the problem and fundamental equations

We consider a homogeneous isotropic, thermally conducting elastic solid, at uniform absolute temperature T_0, in the undisturbed state and discuss the thermal and elastic

plane wave motion of small amplitude. We consider a fixed rectangular Cartesian-coordinate system (x, y, z) with origin on the plane surface $(z = 0)$ and negative z-axis pointing normally into the medium, which is thus represented by $z = 0$. A rotational wave propagating from infinity within the solid is assumed to be incident on the boundary $z = 0$, making an angle θ with the negative direction of z-axis. We also assume that the body is thermally conducting and the thermal wave velocity is small in compared with the dilatational elastic wave velocity.

Due to the application of the initial magnetic field $\mathbf{H}_0 = H_0(0, 1, 0)$ there results an induced magnetic field $\mathbf{h} = h(0, 1, 0)$ and an induced electric field $\mathbf{E}$. The simplified linear equations of electrodynamics of a slowly moving medium for a homogeneous, thermally and electrically conducting elastic solid are

$$curl\,\mathbf{h} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.1a}$$

$$curl\,\mathbf{E} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t}, \tag{2.1b}$$

$$div\,\mathbf{h} = 0, \tag{2.1c}$$

$$\mathbf{E} = -\mu_0 \left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H} \right), \tag{2.1d}$$

where $\mathbf{H}$ is the magnetic field vector, $\mathbf{J}$ the current density vector, μ_0 the magnetic permeability, ϵ_0 the electric permeability and $\mathbf{u}$ the displacement vector.

In the absence with the body force and no heating source, the generalized electro-magneto-thermoelastic governing differential equations are:

Strain-displacement relations:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.2}$$

Duhamel-Neumann constitutive equations:

$$\sigma_{ij} = \lambda\,div(\mathbf{u})\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \gamma T \delta_{ij}. \tag{2.3}$$

Equation of motion, (when the Lorentz electromagnetic force $\mu_0(\mathbf{J} \times \mathbf{H})$ is the body force), has the form

$$\sigma_{ij,j} + \mu_0(\mathbf{J} \times \mathbf{H}) = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{2.4}$$

Keeping all the preceding in mind, we arrive at the fundamental field equation of electromagneto-thermoelasticity in generalized thermoelasticity in its linearized form by eliminating the stress tensor from Eq. (2.4), through the use of Eq. (2.3), this gives

$$(\lambda + \mu)grad(div(\mathbf{u})) + \mu\nabla^2(\mathbf{u}) - \gamma\,grad(T) + \mu_0(\mathbf{J} \times \mathbf{H}) = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{2.5}$$

The Tzou model is such a modified of classical thermoelasticity model in which the Fourier law is replaced by an approximation of the equation

$$q(x, t + \tau_q) = -K\nabla(x, t + \tau_\theta). \tag{2.6}$$

The model transmits thermoelastic disturbances in a wave- like manner (1986) if Eq. (2.6) is approximated by

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right) q_i = -K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right) \nabla T, \tag{2.7}$$

where $0 \leq \tau_\theta < \tau_q$.

Then the heat conduction equation in the context of dual-phase-lage thermoelasticity proposed by Tzou in this case takes the form

$$K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right) \nabla^2 T = \left(1 + \tau_q \frac{\partial}{\partial t}\right) \left(\rho C_E \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial e}{\partial t}\right), \tag{2.8}$$

where σ_{ij} is the stress tensor, λ and μ are the Lamé constants, γ is equal to $\alpha_0(3\lambda + 2\mu)$, α_0 is he coefficient of volume expansion, K is the thermal conductivity, C_E is the specific heat per unit mass at constant strain, ρ is the density of the medium, q_i is the heat flux vector, $e = e_{kk}$ strain dilatation, τ_θ is the phase-lag of the heat flux and τ_q is phase-lag of gradient of temperature.

From the preceding equations, we can get the following thermoelasticity theories:

- The dynamic coupled theory (**C-D** theory) $\tau_\theta = \tau_q = 0$.
- The generalized thermoelasticity theory with one relaxation time (**L-S** theory) $\tau_\theta = 0$, $\tau_q = t_0$, where $t_0 > 0$ is the first relaxation time.
- The generalized thermoelasticity theory with dual-phase-lags (**DPL** model) $\tau_q \geq \tau_\theta > 0$.

3   Solution of the problem

We take the plane wave motion in the xz-plane and $\partial/\partial y = 0$, the displacement components have the following form

$$u_x = u(x, z, t), \quad v_y = 0, \quad w_z = w(x, z, t). \tag{3.1}$$

From Eqs. (2.2) and (3.1), we obtain the strain components

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{zz} = \frac{\partial w}{\partial z}, \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\right),$$

$$e_{yy} = e_{xy} = e_{yz} = 0, \qquad e = div(\mathbf{u}) = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}.$$

We study the reflection of the thermoelastic waves at a stress-free surface, and discuss P and SV waves. We define the angle of incidence θ as the angle between the propagation of plane wave and the normal to the boundary.

Then Eqs. (2.1b) and (2.1d) give

$$\mathbf{E} = \mu_0 H_0 \left(\frac{\partial w}{\partial t}, 0, -u\right), \tag{3.2a}$$

$$\mathbf{h} = -H_0(0, e, 0). \tag{3.2b}$$

Eq. (2.1a) after using relations (3.2a) and (3.2b) yields

$$\mathbf{J} = H_0\Big(\frac{\partial e}{\partial z} - \epsilon_0\mu_0\frac{\partial^2 w}{\partial t^2}, 0, -\frac{\partial e}{\partial x} + \epsilon_0\mu_0\frac{\partial^2 u}{\partial t^2}\Big). \tag{3.3}$$

From Eqs. (3.2b) and (3.3), the Lorentz force take the form

$$\mu_0\Big(\mathbf{J} \times \mathbf{H}\Big) = \mu_0 H_0^2\Big(\frac{\partial e}{\partial x} - \epsilon_0\mu_0\frac{\partial^2 u}{\partial t^2}, 0, \frac{\partial e}{\partial z} - \epsilon_0\mu_0\frac{\partial^2 w}{\partial t^2}\Big). \tag{3.4}$$

To separate the dilatational and rotational components of strain, we introduce the elastic displacement vector into its potential and rotational parts

$$\mathbf{u} = grad(\phi) + curl(\boldsymbol{\psi}), \qquad \boldsymbol{\psi} = \psi\mathbf{e}_2, \tag{3.5}$$

where $\mathbf{e}_2$ is a unit vector in y-direction, then displacement components of vector $\mathbf{u}$ in x-z plane may be written as

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \qquad w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{3.6}$$

Taking divergence of each term of Eq. (2.5) and using Eq. (3.5), we get the equation for dilatation waves as

$$c_0^2\nabla^2\phi - \Big(\frac{c_3^2}{c^2} + 1\Big)\frac{\partial^2\phi}{\partial t^2} = \frac{\gamma}{\rho}T. \tag{3.7}$$

Taking curl of each term in Eq. (2.5) and using some well-known vector identities, we get in a similar way, the equation for shear waves as

$$c_2^2\nabla^2\psi - \Big(\frac{c_3^2}{c^2} + 1\Big)\frac{\partial^2\psi}{\partial t^2} = 0, \tag{3.8}$$

where

$$c_1^2 = \frac{(\lambda + 2\mu)}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad c_3^2 = \frac{\mu_0 H_0^2}{\rho}, \quad c_0^2 = c_1^2 + c_3^2, \quad C_L = \frac{1}{\sqrt{\mu_0\epsilon_0}}.$$

c_1, c_2 are, respectively, the longitudinal and shear wave velocities in the solid c_3 and is magnetic pressure number.

From Eqs. (3.7) and (3.8), we observe that while the P wave is affected by the presence of the thermal wave the SV wave remains unaffected. The solution of Eq. (3.8) corresponds to the propagation of SV wave with velocity c_2. We assume that the modified P and SV waves are independent of each other as in the ordinary case and such we think of modified P wave and SV wave as being separately.

Eliminating T from (2.8) and (3.7), we find that both ϕ satisfy the differential equation

$$\overline{K}\Big(1 + \tau_\theta\frac{\partial}{\partial t}\Big)\nabla^4\phi - \Big[\frac{\overline{K}}{c_4^2}\Big(1 + \tau_\theta\frac{\partial}{\partial t}\Big)\frac{\partial^2}{\partial t^2} + (1 + \epsilon)\Big(\frac{\partial}{\partial t} + \tau_q\frac{\partial^2}{\partial t^2}\Big)\Big]\nabla^2\phi$$

$$+ \frac{1}{c_4^2}\Big(\frac{\partial}{\partial t} + \tau_q\frac{\partial^2}{\partial t^2}\Big)\frac{\partial^2\phi}{\partial t^2} = 0, \tag{3.9}$$

where

$$c_4^2 = \frac{c_0^2}{\beta^2}, \qquad \beta^2 = \left(\frac{c_3^2}{C_L^2} + 1\right), \qquad \epsilon = \frac{\gamma^2 T_0}{\rho^2 c_0^2 C_E}.$$

In order to solve the problem we assume the solution of (3.9) in the form

$$\phi = f(z) \exp\left[i\zeta(x - ct)\right], \tag{3.10}$$

where ζ is the wave number and phase velocity c is equal to $c_1 \sin\theta_0$, c_1 is the velocity of incident wave and θ_0 denotes the angle of incidence.

With the help of (3.10), Eq. (3.9) reduces to the following fourth order differential equation in $f(z)$ in the form

$$\frac{d^4 f(z)}{dz^4} + A \frac{d^2 f(z)}{dz^2} + B f(z) = 0, \tag{3.11}$$

where

$$A = \zeta^2 \left[\left(2 - \frac{c^2}{c_4^2}\right) - \frac{(1+\epsilon)c^2}{c_4^2}(t_q - t_\theta)\right] - \frac{ic\zeta(1+\epsilon)}{\overline{K}}, \tag{3.12a}$$

$$B = \zeta^4 \left[\left(1 - \frac{c^2}{c_4^2}\right) + \frac{c^2\left(1 + \epsilon - \frac{c^2}{c_4^2}\right)}{c_4^2}(t_q - t_\theta)\right] + \frac{ic\zeta^3\left(1 + \epsilon - \frac{c^2}{c_4^2}\right)}{\overline{K}}, \tag{3.12b}$$

where t_q and t_θ are the dimensionless phase-lags given by

$$t_i = \frac{c_4^2}{\overline{K}}\tau_i, \quad i = q, \theta.$$

Therefore, the complete solution for $f(z)$ from Eq. (3.11) is

$$f(z) = A_1 \exp(m_1 z) + A_2 \exp(-m_1 z) + A_3 \exp(m_2 z) + A_4 \exp(-m_2 z), \tag{3.13}$$

where

$$m_1 = \left[\frac{\sqrt{A^2 - 4B} - A}{2}\right]^{\frac{1}{2}}, \qquad m_2 = \left[\frac{-\sqrt{A^2 - 4B} - A}{2}\right]^{\frac{1}{2}}.$$

Thus we see that while the root m_1 corresponds to a thermal wave, m_2 corresponds to a modified longitudinal P-wave and A_i, $(i = 1, 2, 3, 4)$ are pairs represent the amplitudes of incident and reflected thermal, P- and SV-waves, respectively.

By the same method we assume the solution of (3.8) in the form

$$\psi = g(z) \exp\left[i\zeta(x - ct)\right], \tag{3.14}$$

we find the solution of ψ can be written as

$$\psi = \{A_5 \exp(i\zeta bz) + A_6 \exp(-i\zeta bz)\} \exp\left[i\zeta(x - ct)\right], \tag{3.15}$$

where

$$b = \sqrt{\frac{c^2}{c_5^2} - 1}, \qquad c_5^2 = \frac{c_2^2}{\beta^2},$$

correspond to the transverse wave and A_5, A_6 are arbitrary constants.

Introducing Eqs. (3.10) and (3.13) into Eq. (3.7), we obtain

$$T = \frac{\rho\beta^2}{\gamma}\big[b_1 A_1 \exp(m_1 z) + b_1 A_2 \exp(-m_1 z) + b_2 A_3 \exp(m_2 z)$$
$$+ b_2 A_4 \exp(-m_2 z)\big] \exp\big[i\zeta(x - ct)\big], \qquad (3.16)$$

where

$$b_i = c_4^2\Big[m_i^2 + \zeta^2\Big(\frac{c^2}{c_4^2} - 1\Big)\Big], \quad i = 1, 2. \qquad (3.17)$$

Since the temperature T should be non-infinite at $z \to \infty$, we take $A_2 = 0$. Then the expressions of $\phi, T, u, w, \sigma_{zz}$ and σ_{xz} will be in the forms (omitting the exponential term $\exp[i\zeta(x - ct)]$)

$$\phi = A_1 \exp(m_1 z) + A_3 \exp(m_2 z) + A_4 \exp(-m_2 z), \qquad (3.18a)$$

$$T = \frac{\rho\beta^2}{\gamma}\big[b_1 A_1 \exp(m_1 z) + b_2 A_3 \exp(m_2 z) + b_2 A_4 \exp(-m_2 z)\big], \qquad (3.18b)$$

$$u = i\zeta\big[A_1 \exp(m_1 z) + A_3 \exp(m_2 z) + A_4 \exp(-m_2 z)$$
$$- b A_5 \exp(i\zeta b z) + b A_6 \exp(-i\zeta b z)\big], \qquad (3.18c)$$

$$w = \big[m_1 A_1 \exp(m_1 z) + m_2 A_3 \exp(m_2 z) - m_2 A_4 \exp(-m_2 z)$$
$$+ i\zeta A_5 \exp(i\zeta b z) + i\zeta A_6 \exp(-i\zeta b z)\big], \qquad (3.18d)$$

$$\sigma_{zz} = \rho c_2^2 \zeta^2\big[q_1 A_1 \exp(m_1 z) + q_2 A_3 \exp(m_2 z) + q_2 A_4 \exp(-m_2 z)$$
$$- 2b A_5 \exp(i\zeta b z) + 2b A_6 \exp(-i\zeta b z)\big], \qquad (3.18e)$$

$$\sigma_{xz} = i\zeta\rho c_2^2\big[2\big(m_1 A_1 \exp(m_1 z) + m_2 A_3 \exp(m_2 z) - m_2 A_4 \exp(-m_2 z)\big)$$
$$+ i\zeta(1 - b^2)\big(A_5 \exp(i\zeta b z) + A_6 \exp(-i\zeta b z)\big)\big], \qquad (3.18f)$$

where

$$q_1 = \frac{c_3^2}{c_2^2}\Big(1 - \frac{m_1^2}{\zeta^2}\Big) + \Big(2 - \frac{\beta^2 c^2}{c_5^2}\Big), \qquad q_2 = \frac{c_3^2}{c_2^2}\Big(1 - \frac{m_2^2}{\zeta^2}\Big) + \Big(2 - \frac{\beta^2 c^2}{c_5^2}\Big).$$

4 Boundary conditions

The surface $z = 0$ is assumed to be traction free and thermally insulated so that there is no variation of temperature on it. Therefore, the boundary conditions on $z = 0$ are written as

$$\sigma_{zz} = 0, \quad \sigma_{xz} = 0, \quad \frac{\partial T}{\partial z} = 0, \quad \text{at } z = 0. \qquad (4.1)$$

Using the potentials given by Eqs. (3.18b), (3.18e) and (3.18f) in boundary conditions (4.1), we get

$$q_1 A_1 + q_2 A_3 + q_2 A_4 - 2bA_5 + 2bA_6 = 0, \tag{4.2a}$$

$$m_1 A_1 + m_2 A_3 - m_2 A_4 + a_1 A_5 + a_1 A_6 = 0, \tag{4.2b}$$

$$m_1 b_1 A_1 + b_2 m_2 A_3 - m_2 b_2 A_4 = 0, \tag{4.2c}$$

where $a_1 = i\zeta(1 - b^2)/2$.

Since we have five unknown and only three equation, we consider the incident wave to be either a P wave or SV wave.

(I) **In the case of incident P wave**

In the case of incident P wave, we take $A_5 = 0$. Since $b_1 \gg b_2$, the solution of Eqs. (4.2a)-(4.2c) will be

$$\frac{A_4}{A_3} = \frac{-b_1 a_1 m_1 q_2 + 2m_1 c_2^2 bb_1 m_2 + q_1 a_1 b_2 m_2 - 2m_1 c_2^2 bb_2 m_2}{b_1 a_1 m_1 q_2 - 2m_1 c_2^2 bb_2 m_2 + 2m_1 c_2^2 bb_1 m_2 + q_1 a_1 b_2 m_2} \approx \frac{-aq_2 + 2c_2^2 bm_2}{aq_2 + 2c_2^2 bm_2}, \tag{4.3a}$$

$$\frac{A_6}{A_3} = \frac{-2m_2 m_1 q_2(-b_2 + b_1)}{b_1 a_1 m_1 q_2 - 2m_1 c_2^2 bb_2 m_2 + 2m_1 c_2^2 bb_1 m_2 + q_1 a_1 b_2 m_2} \approx \frac{-2m_2 q_2}{a_1 q_2 + 2c_2^2 bm_2}, \tag{4.3b}$$

$$\frac{A_1}{A_3} = \frac{-2m_2 a_1 q_2 b_2}{b_1 a_1 m_1 q_2 - 2m_1 c_2^2 bb_2 m_2 + 2m_1 c_2^2 bb_1 m_2 + q_1 a_1 b_2 m_2} \approx 0. \tag{4.3c}$$

For the above values of the relevant physical constants, the systems of Eqs. (4.2a)-(4.2c) are solved for amplitude ratios by the application of the Gauss elimination method.

(II) **In the case of incident SV wave**

In the case of incident SV wave, we take $A_3 = 0$, then the solution of Eqs. (4.2a)-(4.2c) take the forms

$$\frac{A_4}{A_5} = \frac{4a_1 b_1 m_1 c_2^2 b}{q_2 b_1 m_1 a_1 + 2b_1 m_1 c_2^2 b - 2m_2 b_2 c_2^2 b + m_2 b_2 q_1 a_1} \approx \frac{4a_1 c_2^2 b}{q_2 a_1 + 2c_2^2 b}, \tag{4.4a}$$

$$\frac{A_6}{A_5} = \frac{2b_1 m_1 c_2^2 b - 2m_2 b_2 c_2^2 b - q_2 b_1 m_1 a_1 - m_2 b_2 q_1 a_1}{q_2 b_1 m_1 a_1 + 2b_1 m_1 c_2^2 b - 2m_2 b_2 c_2^2 b + m_2 b_2 q_1 a_1} \approx \frac{2c_2^2 b - q_2 a_1}{q_2 a_1 + 2c_2^2 b}, \tag{4.4b}$$

$$\frac{A_1}{A_5} = \frac{4m_2 b_2 a_1 c_2^2 b}{q_2 b_1 m_1 a_1 + 2b_1 m_1 c_2^2 b - 2m_2 b_2 c_2^2 b + m_2 b_2 q_1 a_1} \approx 0. \tag{4.4c}$$

5 Reflection and partition of energy at a free surface

Now let us consider (see figure below) a beam of incident of cross-sectional area ΔS_0. The corresponding beams of reflected P-waves and SV-waves are of cross-sectional areas ΔS_1 and ΔS_2, respectively. Since the surface area ΔS is free of tractions and since no energy is dissipated, the average energy transmission across ΔS_0 must equal the sum of the average energy transmissions across ΔS_1 and ΔS_2.
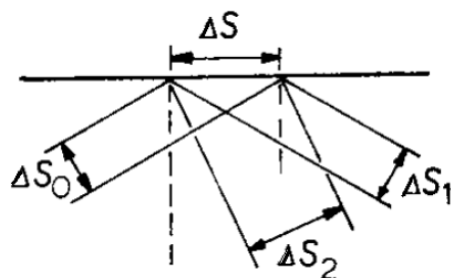
Beams of incident and reflected waves.

Writing the equality between incident P-wave and thermal wave energy and reflected P-wave and SV-waves energy per unit area of the surface, we get in the case of incident P-wave [8]

$$\left(\frac{A_4}{A_3}\right)^2 + \left(\frac{ikb}{m_2}\right)\left(\frac{A_6}{A_3}\right)^2 = 1, \tag{5.1}$$

and in the case of incident SV-wave

$$\left(\frac{m_2}{ikb}\right)\left(\frac{A_4}{A_5}\right)^2 + \left(\frac{A_6}{A_5}\right)^2 = 1. \tag{5.2}$$

It can be checked that the previously derived amplitude ratios satisfy Eqs. (5.1) and (5.2). From Eqs. (5.1) and (5.2) we can also determine how the average energy transmission is partitioned over the reflected P-wave and the reflected SV-wave.

6   Numerical example

For computational work, to illustrate the analytical procedure presented earlier, we consider now a numerical example. The results depict the variation for the partition of energy with various values of the angle of incidence θ_0. For this purpose, sand stone is considered as the thermoelastic material body for which we have the physical constants as follows [8].

$$\rho = 2.30 \text{g}/(\text{cm})^3, \quad \alpha_0 = 0.4 \times 10^{-5}/(\text{degC}), \quad \lambda = \mu = 0.8 \times 10^{11} \text{dyne (cm)}^2,$$
$$H_0 = 10^6 \text{Oe}, \qquad C_E = 0.23 \text{cal/gm degC}, \qquad K = 0.6 \times 10^{-2} \text{cal/cm sec degC}.$$

Depending on these constants, we find

$$m_1 = \frac{\sqrt{i(Re(A) + 2iIm(A))}}{\sqrt{2}}, \qquad m_2 = \frac{i}{\sqrt{2}}\sqrt{i(Re(A))},$$

where $Re(A)$ and $Im(A)$ represents the real and imaginary parts of A respectively.

The variations of the partition of energy

$$\left(\left(\frac{A_4}{A_3}\right)^2, \left(\frac{ikb}{m_2}\right)\left(\frac{A_6}{A_3}\right)^2\right) \quad \text{and} \quad \left(\left(\frac{A_6}{A_5}\right)^2, \left(\frac{m_2}{ikb}\right)\left(\frac{A_4}{A_5}\right)^2\right),$$
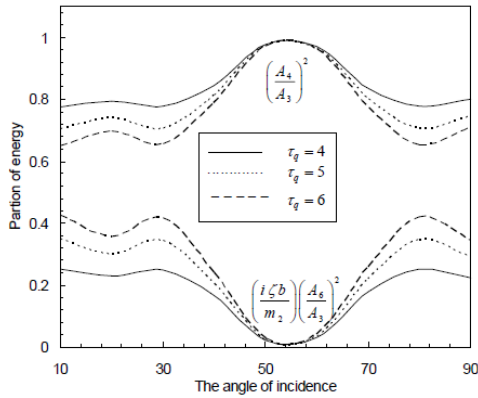
Figure 1: The variations of partition of energy of P wave with the angle of incidence θ_0 for phase-lag of the heat flux $\tau_q = 4, 5, 6$.
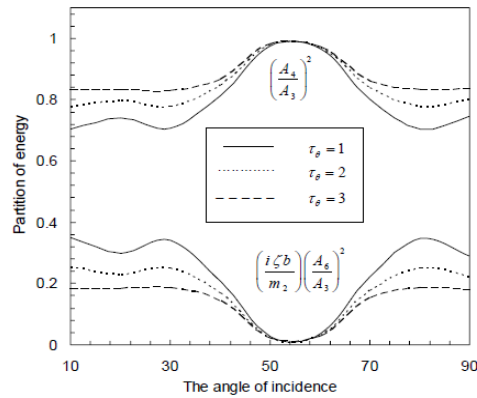

Figure 2: The variations of partition of energy of P wave with the angle of incidence θ_0 for phase-lag of temperature gradient $\tau_\theta = 1, 2, 3$.
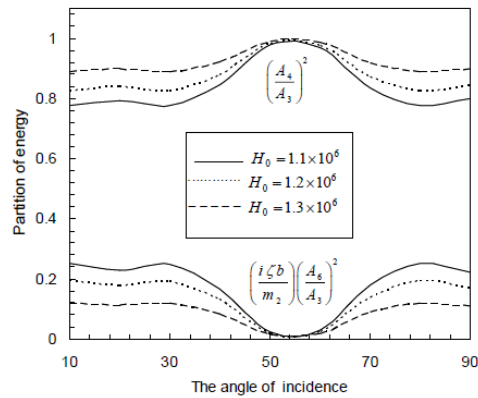

Figure 3: The variations of partition of energy of P wave with the angle of incidence θ_0 for magnetic $H_0 = (1.1, 1.2, 1.3) \times 10^6$.
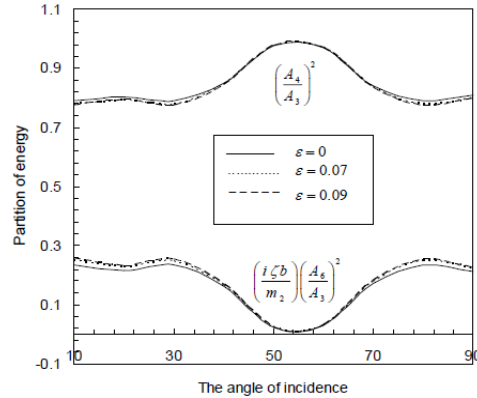

Figure 4: The variations of partition of energy of P wave with the angle of incidence θ_0 for coupling parameter $\varepsilon = 0, 0.07, 0.09$.

for incident P and SV-waves have been computed for various values of the angle of incidence θ_0 and have been shown graphically in Figs. 1-8.

Figs. 1 and 5 give the effect of the phase-lag of the heat flux τ_q (with fixed value of $\tau_\theta = 2$), however Figs. 2 and 6 give the effect of the phase-lag of temperature gradient τ_θ ($\tau_q = 4$) with the angle of incidence on the reflection coefficient ratios. It can be noticed that:

1. The reflection coefficients $(A_4/A_3)^2$ and $(A_6/A_5)^2$ are decreasing with the increase in the values of τ_q and τ_θ while $(ikb/m_2)(A_6/A_3)^2$ and $(m_2/ikb)(A_4/A_5)^2$ are increasing in the case of P and SV waves.

2. $(A_4/A_3)^2$ and $(A_6/A_5)^2$ increase monotonically in $0° \leq \theta_0 \leq 54°$ and become unity at $54°$ and decreases in the range $54° \leq \theta_0 \leq 90°$ for P and SV waves.

3. $(ikb/m_2)(A_6/A_3)^2$ and $(m_2/ikb)(A_4/A_5)^2$ decrease monotonically to become zero in the range $0° \leq \theta_0 \leq 54°$ and attain their minimum values at approximately $54°$

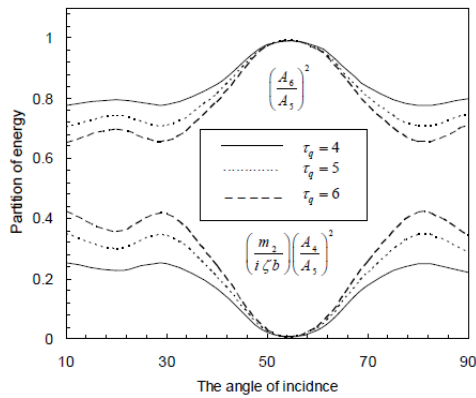Figure 5: The variations of partition of energy of SV wave with the angle of incidence θ_0 for phase-lag of the heat flux $\tau_q = 4, 5, 6$.
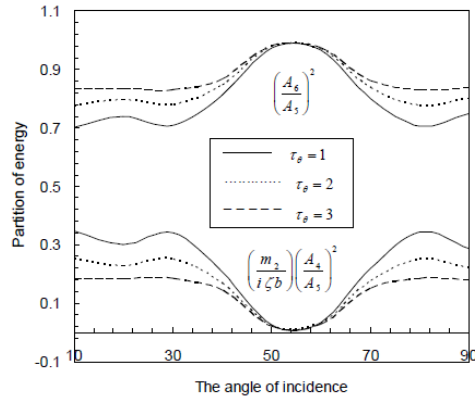

Figure 6: The variations of partition of energy of SV wave with the angle of incidence θ_0 for phase-lag of temperature gradient $\tau_\theta = 1, 2, 3$.
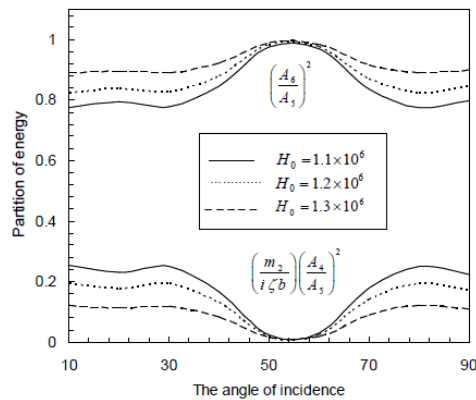

Figure 7: The variations of partition of energy of SV wave with the angle of incidence θ_0 for magnetic $H_0 = (1.1, 1.2, 1.3) \times 10^6$.
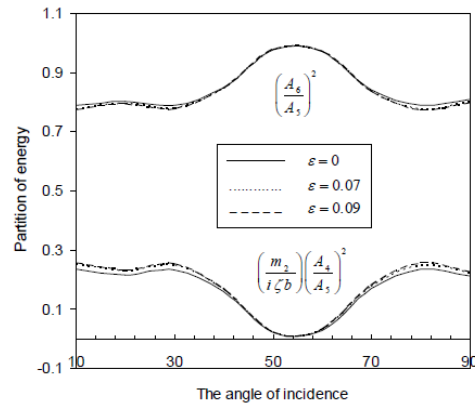

Figure 8: The variations of partition of energy of SV wave with the angle of incidence θ_0 for coupling parameter $\varepsilon = 0, 0.07, 0.09$.

and then decreases in the range $54° \leq \theta_0 \leq 90°$ for P and SV waves.

4. The sum of energy ratios is unity at each angle of incidence for various reflected waves.

In Figs. 3 and 7, we observe that the magnetic field has a small influence on the reflection coefficients ratio. Figs. 4 and 8 give the variations of the angle of incidence with the reflection coefficient ratios against different values of coupling parameter ε for the incident waves. We see that

1. The reflection coefficients $(A_4/A_3)^2$ and $(A_6/A_5)^2$ are increasing under the effect of the magnetic field for P- and SV-waves. On the other hand, the reflection coefficients $(ikb/m_2)(A_6/A_3)^2$ and $(m_2/ikb)(A_4/A_5)^2$ are decreasing in the case of P- and SV-waves.

2. With the increase in the values of ε the reflection coefficients $(A_4/A_3)^2$ and $(A_6/A_5)^2$ are decreasing but $(ikb/m_2)(A_6/A_3)^2$ and $(m_2/ikb)(A_4/A_5)^2$ are increasing in the case of P- and SV-waves.

3. The effect of the coupling parameter and the applied magnetic field on the waves can be clearly observed in the figures.

7 Conclusions

In this paper, the effect of the dual-phase-lag **(DPL)** model and magnetic field on the reflection of thermoelastic waves at a half space is studied. Numerical computations have been performed for a particular model and the variations of amplitude and energy ratios are obtained against the angle of incidence. The results obtained are depicted graphically. It has been shown that the sum of energy ratios is unity at the interface. It is shown that the amplitude ratios of reflected and transmitted waves depend on the angle of incidence, frequency and elastic properties of the media.

It can be concluded that

- The partition of energy depends on the angle of incidence and material constants.
- While the P wave is affected by the presence of the thermal wave, the SV wave remains unaffected.
- The effects of the phase-lag of the heat flux τ_q and the phase-lag of the temperature gradient τ_θ on the reflection coefficient ratios are significantly observed.
- The magnetic field has a significant effect on the reflection coefficient ratios.
- The thermal coupling parameter has a small influence on the reflection coefficient ratios.

The final results also indicate that the problems of waves become more important in the field of geology, when we study the problem with additional parameters (e.g., thermal disturbance, magnetic fields, etc). It may represent a more realistic form of the earth model and may be of the interest for experimental seismologist. Moreover, it is employed shallow application for engineering, groundwater and environmental surveying.

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