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Equivalent a Posteriori Error Estimator of Spectral Approximation for Control Problems with Integral Control-State Constraints in One Dimension

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Abstract. In this paper, we investigate the Galerkin spectral approximation for elliptic control problems with integral control and state constraints. Firstly, an a posteriori error estimator is established, which can be acted as the equivalent indicator with explicit expression. Secondly, appropriate base functions of the discrete spaces make it is probable to solve the discrete system. Numerical test indicates the reliability and efficiency of the estimator, and shows the proposed method is competitive for this class of control problems. These discussions can certainly be extended to two- and three-dimensional cases.

AMS subject classifications: 49J20, 65K15, 65M12, 65M70

Key words: Optimal control, elliptic equations, control-state constraints, spectral method, a posteriori error estimator.

1 Introduction

In this paper, we discuss the following control problem:

$$\begin{cases} \min J(u,y) = \dfrac{1}{2}\|y(u)-y_0\|_{0,I}^2 + \dfrac{\alpha}{2}\|u\|_{0,I}^2, \\ \text{Subject to } \; -y''(u) = u+f \;\text{ in }\; I, \quad y(u)=0 \;\text{ on }\; \partial I, \\ (u,y)\in U_{ad}\times K, \end{cases} \tag{1.1}$$

where U_{ad} and K are closed convex sets, which can represent the optimal control problems with control, state, or control-state constraints. Both control- and state-constrained cases can be met frequently in real-life applications and engineering designs, and there have been lots of results on both theoretical aspects and numerical approximation for

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them. In respect of the theoretical analysis, extensive research has been carried out on optimality conditions, regularity of approximated solutions, the existence of optimal control and Lagrange multipliers, see, for example, [2,4,13,17,18,22,24]. On the numerical methods, one can find a large number of valuable works. The authors gave a detailed introduction for adaptive finite element approximation of optimal control problems with control constraints governed by PDEs in [24]. The primal-dual active set algorithm was used to approximate state-constrained problems in [1], and an augmented Lagrangian method was investigated to solve state and control constrained problems in [3]. A semi-smooth Newton method was proposed to compute state-constrained control problems in [21]. In many cases, we need place restrictions on average values or energy-norms of the state or control variables. The authors derived a priori error estimates and equivalent residual-type a posteriori error estimator of finite element approximation for control problems governed by Stokes equations with L^2 state constraint in [27] and [28]. A gradient projection algorithm was used to solve control problems with integral constraint for state in [23,37].

As to the control-state constrained problems, the existence of Lagrange multipliers was discussed in [6,32,36], and the optimality conditions were derived in [6,30,31]. In respect of numerical approximation, A posteriori error estimator that contains only computable quantities was provided for finite element approximation of elliptic control problems in [33]. The reliability and efficiency of the residual type a posteriori error estimator were proved for the finite element approximation of the control problems in [20]. The elliptic optimal control problems were approximated by finite element method in [10], and the suitable relation between the regularization parameter and the mesh size can be helpful for deriving a good convergence order. The control problems were reformulated into a nonlinear programming, which can be solved by several optimization codes in [26]. A primal interior point method was applied to control problems with mixed control-state constraints in [29]. However, most of these works focused on the optimal control with pointwise constraints.

Thanks to the fast convergence rate, high-order accuracy, and a small amount of unknowns, the spectral method has been successfully used in numerical solutions of PDEs and fluid dynamics (see, for example, [7,9,35]). Recently, researches investigated the spectral approximation of optimal control problems governed by PDEs. In [11,12], the Galerkin spectral method was used to solve the control problems with integral control constraint, where both a priori error estimates and a posteriori error estimates were derived. The efficiency of the proposed methods were indicated by numerical tests, which confirmed the convergence results of a priori error estimates. In [38], spectral method was studied to approximate state constrained control problems governed by the first biharmonic equation, and a priori error estimates were derived. We can see that the spectral method has gained increasing popularity in approximating control problems in the last decade. However, as an upper bound, the a posteriori error estimators in [11,12] were difficult to evaluate, and the numerical examples didn't illustrate the performance of them. In this paper, the use of one-dimensional space allows us to focus on the techniques for