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A Combined Discontinuous Galerkin Method for Saltwater Intrusion Problem with Splitting Mixed Procedure

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Abstract. In this paper, a new combined method is presented to simulate saltwater intrusion problem. A splitting positive definite mixed element method is used to solve the water head equation, and a symmetric discontinuous Galerkin (DG) finite element method is used to solve the concentration equation. The introduction of these two numerical methods not only makes the coefficient matrixes symmetric positive definite, but also does well with the discontinuous problem. The convergence of this method is considered and the optimal L^2-norm error estimate is also derived.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Splitting mixed system, discontinuous Galerkin method, saltwater intrusion problem, convergence analysis.

1 Introduction

In recent years, saltwater intrusion has occurred in many countries and regions all over the world, and caused great damages to industrial and agricultural productions. In order to prevent and control this problem, more and more work on the problem has been done in past decades. This problem is often described by coupled density-dependent groundwater flow and advection-dispersion equations because of hydrodynamic dispersion and a wide transition zone [8, 12, 14, 23]. Numerical methods for solving the saltwater intrusion problem include the method of characteristics (MOC) [18, 19, 30], the finite element method (FEM) [22, 26] and the finite difference method (FDM) [11, 29], and so on.

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However, using the traditional finite element or finite difference method to solve the water head equation, we can not directly obtain the approximate flux which appears in the concentration equation. And differentiating the water head function to obtain the flux will result in extra error and reduce the accuracy. To obtain more accurate approximation of the flux function, the classical mixed finite element method [15] and the mixed hybrid finite element method (MHFE) [2,6] were introduced. But these techniques result in some saddle point problems whose numerical solutions have been quite difficult because of losing positive definite properties. In [28, 32–35], the authors of this paper proposed a class of splitting mixed finite element methods, in which the mixed system is symmetric positive definite and the flux equation is separated from the original equation.

Moreover, the concentration equation is parabolic and normally convection-dominated. The standard Galerkin method applied to the convection-dominated problems does not work well, and produces excessive numerical diffusion or nonphysical oscillation. A variety of numerical techniques have been introduced to obtain better approximations, such as higher-order Godunov method [4], streamline diffusion method [13], least-squares method [27], and the characteristic finite element methods [31,38,39]. However, these methods mentioned above are all continuous and can not deal with the discontinuous case. Discontinuous Galerkin finite element (DGFE) methods were introduced independently in [5,10,20]. Since then, they have become more and more powerful tools to deal with many types of discontinuous problems [3,21,24,25]. A combined classical mixed element scheme with DGFE method for incompressible miscible displacement problems was presented in [24,25]. The similar method was extended in [9] to solve the compressible miscible displacement problem. The DGFE techniques have been applied by the authors of this paper [16,17,36,37], to nonlinear partial differential equations.

The main purpose of this paper is to construct a combined numerical method for solving saltwater intrusion problem: the splitting mixed element method is used to solve the water head equation and the symmetric interior penalty discontinuous Galerkin method is used to solve the concentration equation. The application of the splitting mixed element method results in a symmetric positive definite coefficient matrix of the mixed element system and separating the flux equation from the water head equation so that one can obtain an approximate solution of the flux function fast and independently by using various effective algorithms. Meantime, the discontinuous Galerkin method does well in handling the discontinuous areas since the trial and test functions across the finite element interfaces do not explicitly impose continuity constraints. In order to derive the optimal error estimate in L^2-norm, the classical mixed finite element spaces, which is not necessary in practice, will be used and a special projection of the concentration c, which imposes much additional condition to the penalty, is also used. In this paper, the restriction on the penalty is not our main work since it had been considered.

In order to illustrate our method, the following mathematical model of saltwater intrusion problem is considered: a coupled system composed of the water head equation and the concentration (of Cl^-) equation