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Fibre-Reinforced Generalized Anisotropic Thick Plate with Initial Stress under the Influence of Fractional Thermoelasticity Theory

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Abstract. In the present work concentrated on the two-dimensional problem of generalized thermoelasticity for a fiber-reinforced anisotropic thick plate under initial stress. Using generalized thermoelasticity theory with fractional order heat conduction, the problem has been solved by a normal mode analysis. The effect of hydrostatic initial stresses and fractional order parameter is shown graphically on the distributions of the temperature, displacement and thermal stress components. It is found from the graphs that the initial stress and the fractional parameter significantly influences the varieties of field amounts.

AMS subject classifications: 74K25, 74G60

Key words: Fiber-reinforced, anisotropy, fractional thermoelasticity initial stress, thick plate.

1 Introduction

In the recent, consideration has been given to the problems of generation and propagation of elastic waves in an anisotropic elastic solids or layers of various configurations as the propagation of elastic waves in anisotropic media is fundamentally not the same as their propagation in isotropic media. The data obtained from such study is essential to seismologists and geophysicists to find the location of the earthquakes and additionally their energy, mechanism etc. and thereby gives vsignificant knowledge into the global tectonics. Accessible data recommends that the layered media, crystals and different materials, for example, fiber reinforced materials, fluid saturated porous materials etc. exhibits anisotropy. Some hard and soft rocks underneath the earth surface show the

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reinforcement properties, i.e., the diverse components goes about as a single anisotropic unit. These rocks when come in the method of seismic waves do influence their spread and such seismic signals are always influenced by the elastic properties of the media through which they travel.

The mechanical behavior of many fibre-reinforced composite materials is sufficiently shown by the theory of linear elasticity for transversely isotropic materials, with the favored direction coinciding with the fibre direction. In such composites the fibres are usually arranged in parallel straight lines. Nonetheless, different designs are utilized. An illustration is that of circumferential reinforcement, for which the fibres are arranged in concentric circles, giving strength and stiffness in the tangential (or hoop) direction. Fibre-reinforced composites are utilized as a part of an assortment of structures because of their low weight and high strength. A continuum model is utilized to clarify the mechanical properties of such materials. On account of an elastic solid reinforced by a series of parallel fibres it is usual to assume transverse isotropy. In the linear case, the related constitutive relations, relating infinitesimal stress and strain components, have five materials constants. The investigation of stress and deformation of fibre-reinforced composite materials has been an imperative subject of solid mechanics for most recent three decades.

The idea of presenting a continuous self reinforcement at each point of elastic solids was given by Belfied et al. [1]. Verma and Rana [2] applied this model to the rotation of a tube. Sengupta and Nath [3] investigated a problem of the surface waves in fiber-reinforced anisotropic elastic media. Hashin and Rosen [4] gave the elastic moduli for fiber-reinforced materials. The problem of reflection of plane waves at the free surface of a fiber-reinforced elastic half-space was discussed by Singh and Singh [5]. Singh [6] discussed the wave propagation in an incompressible transversely isotropic fibre-reinforced elastic media. Singh [7] studied the effects of anisotropy on reflection coefficients of plane waves in fibre-reinforced thermoelastic solid. Kumar and Gupta [8] investigated a source problem in fibre-reinforced anisotropic generalized thermoelastic solid under acoustic fluid layer. Ailawalia and Budhiraja [9] discussed the the effect of hydrostatic initial stress on fibre-reinforced generalized thermoelastic medium. Abbas and Abd-Alla [10] studied the effect of initial stress on a fiberreinforced anisotropic thermoelastic thick plate. Kumar anf Gupta [11] investigated with the propagation of waves in the layer of an anisotropic fibre reinforced thermoelastic solid. Abouelregala and Zenkour [12] studied the effect of rotation on the general model of the equations of the generalized thermoelasticity with fractional order for a homogeneous isotropic elastic half-space solid, whose surface is subjected to a Mode-I crack problem. Abouelregala and Zenkour, [13] investigated the generalized thermoelasticity problem for an infinite fiber-reinforced thick plate under initial stress.

The theory to include the effect of temperature change, known as the theory of thermoelasticity, has also been well established. According to the theory, the temperature field is coupled with the elastic strain field. In thermoelasticity, classical heat transfer, Fourier's conduction equation is extensively used in many engineering applications. The classical theory of thermoelasticity Nowacki, [14, 15] rests upon the hypothesis of the

Fourier law of heat conduction, in which the temperature distribution is governed by a parabolic-type partial differential equation. Consequently, the theory predicts that a thermal signal is felt instantaneously everywhere in a body. This implies that an infinite speed of propagation of the thermal signal, which is impractical from the physical point of view, particularly for short-time. Thus, the use of Fourier's equation may result in discrepancies under some special conditions, such as low-temperature heat transfer, highfrequency or ultrahigh heat flux heat transfer, and so on.

In the classical dynamical coupled theory of thermoelasticity, the thermal and mechanical waves propagate with an infinite velocity, which is not physically admissible. Lord and Shulman [16] introduced a theory of generalized thermoelasticity with one relaxation time for an isotropic body. In this theory, a modified law of heat conduction including both the heat flux and its time derivatives replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both coupled and uncoupled theories of thermoelasticity. Green and Lindsay [17] (referred to as the **GL** theory) extended the coupled theory of thermoelasticity by introducing the thermal relaxation times in the constitutive equations. These theories eliminates the paradox of infinite velocity of heat propagation, are the generalized theories of thermoelasticity. The theory of thermoelasticity without energy dissipation is another generalized theory and was formulated by Green and Naghdi [18]. It includes the thermal displacement gradient among its independent constitutive variables, and differs from the previous theories in that it does not accommodate dissipation of thermal energy [19].

Fractional calculus is a natural extension of the classical mathematics. In fact, since the foundation of the differential calculus the generalization of the concept of derivative and integral to a non-integer order has been the subject of distinct approaches. Due to this reason there are several definitions [20–22] which are proved to be equivalent. Fractional calculus has been applied in many fields, ranging from statistical physics, chemistry to biological sciences and economics. In recent years, there has been a great deal of interest in fractional differential equations. Several definitions of the fractional derivative have been proposed. The history and classic transform rules of this subject are well covered in the monograph by Podlubny [23].

During recent years, fractional calculus has also been introduced in the field of thermoelasticity. Povstenko [24] has constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional order time derivative. He used the Caputo fractional derivative and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. In 2010, a new theory of generalized thermoelasticity in the context of a new consideration of heat conduction with a fractional order has been proposed by Youssef [25]. In the same year, Sherief et al. [26] has constructed another model in generalized thermoelasticity theory by using fractional time derivatives. Abouelregal [27] used the generalized thermoelasticity theory that, based on a fractional order model, to solve a one-dimensional boundary value problem of a semi-infinite piezoelectric medium.

The development of initial stresses in the medium is due to many reasons, for example, resulting from differences of temperature, process of quenching, shot pinning and cold working, slow process of creep, differential external forces, gravity variations, etc. The earth is assumed to be under high initial stresses. It is, therefore, of much interest to study the influence of these stresses on the propagation of stress waves. Biot [28] showed the acoustic propagation under initial stress, which is fundamentally different from that under a stress-free state. He has obtained the velocities of longitudinal and transverse waves along the coordinates axis only.

The wave propagation in solids under initial stresses has been studied by many authors for various models. The study of reflection and refraction phenomena of plane waves in an unbounded medium under initial stresses is due to Chattopadhyay et al. [29], Sidhu and Singh [30] and Dey et al. [31]. Montanaro [32] investigated the isotropic linear thermoelasticity with hydrostatic initial stress. Singh et al. [33], Singh [34] and Othman and Song [35] studied the reflection of thermoelastic waves from a free surface under a hydrostatic initial stress in the context of different theories of generalized thermoelasticity. Ailawalia et al. [36] investigated deformation in a generalized thermoelastic medium with hydrostatic initial stress.

The two-dimensional problem of generalized thermoelasticity for a fiber-reinforced anisotropic thick plate under initial stress is studied in the context of the fractional order theory. The upper surface of the plate is thermally insulated with prescribed surface loading while the lower surface of the plate rests on a rigid foundation and temperature. The problem is solved numerically using a normal mode analysis. Numerical results for the temperature distribution, and the displacement and stress components are given and illustrated graphically. It is found from the graphs that the initial stress significantly influences the variations of field quantities. The results obtained in this paper may offer a theoretical basis and meaningful suggestions for the design of various fiber-reinforced anisotropic thermoelastic elements under loading to meet special engineering requirements.

2 Basic equations of fractional thermoelasticity theory for fiber-reinforced solids

The governing equations for a homogeneous transversely isotropic fiber-reinforced solid with hydrostatic initial stress in the context of generalized thermoelasticity with fractional order without any heat sources or body forces take the following forms:

The equation of motion [9]

$$\sigma_{ij,j}+(u_{i,k}\sigma^0_{kj})_{,j}=\rho\frac{\partial^2 u_i}{\partial t^2}, \tag{2.1}$$

where σ_{ij} are the components of stress, σ^0_{kj} is the initial stress tensor, ρ is the density, u_i

are the components of displacement vector and i,j, $k=1,2,3$. The comma denotes space-coordinate differentiation and the repeated index in the subscript implies summation.

The modified heat conduction equation with fractional order in the absence of heat sources has the form [26]

$$(K_{ij}T_{,i})_{,i}=\Big(\delta+t_0\frac{\partial^\alpha}{\partial t^\alpha}\Big)\Big[\rho C_E\frac{\partial T}{\partial t}+T_0\frac{\partial}{\partial t}(\beta_{ij}u_{i,i})\Big],\quad 0<\alpha\le 1, \tag{2.2}$$

where K_{ij} is the thermal conductivity tensor, t_0 is a constant with the dimensions of time that acts as a relaxation time, C_E is the specific heat at constant strain, T_0 is the temperature of the medium in its natural state, assumed to be such as $|(T-T_0)/T|\ll 1$, β_{ij} is the thermal elastic coupling tensor and e_{ij} are the components of the strain tensor.

The whole spectrum from local heat conduction through the standard heat conduction to the ballistic heat conduction is described by Eq. (2.2). The different values of the parameter α with wide range $(0<\alpha\le 2)$ cover different cases of the conductivity; $(0<\alpha<1)$ correspond to weak conductivity, $\alpha=1$ for normal conductivity, $(1<\alpha<2)$ correspond to strong diffusion conductivity and $\alpha=2$ for to ballistic conductivity.

The constitutive equations for a fibre-reinforced linearly thermoelastic anisotropic medium with respect to the reinforcement direction $\mathbf{b}\equiv(b_1,b_2,b_3)$, where $b_1^2+b_2^2+b_3^2=1$ are written as [9]

$$\begin{aligned}\sigma_{ij}=&\lambda e_{kk}\delta_{ij}+2\mu_T e_{ij}+\chi(b_kb_me_{km}\delta_{ij}+b_ib_je_{kk})\\&+2(\mu_L-\mu_T)(b_kb_ie_{kj}+b_kb_je_{ki})+\beta b_kb_me_{km}b_ib_j-\gamma(T-T_0),\end{aligned} \tag{2.3}$$

where λ, μ_T are the elastic constants, χ, β, $(\mu_L-\mu_T)$ are the reinforcement parameters, and δ_{ij} is Kronecker delta. The comma notation is used for spatial derivatives and superimposed dot represents time differentiation. The theories of coupled thermoelasticity, generalized thermoelasticity with one relaxation time and the generalized theory without energy dissipation follow as limited cases depending on the value of δ, t_0 and α.

The heat conduction Eq. (2.2), in the limiting case $\alpha\to 0$ and $\delta=1$ transforms to:

$$(K_{ij}T_{,i})_{,i}=\Big(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\Big)(\rho C_ET+T_0\beta_{ij}u_{i,i}),$$

which is the same equation obtained by the generalized theory with one relaxation time.

In the limiting case, when $\alpha\to 0$, $t_0=1$ and $\delta=0$, the heat conduction Eq. (2.2), transforms to:

$$(K_{ij}T_{,i})_{,i}=\rho C_E\frac{\partial^2 T}{\partial t^2}+\gamma T_0\frac{\partial^2}{\partial t^2}(\beta_{ij}u_{i,i}),$$

which is the same equation of the generalized theory without energy dissipation introduced by Green and Naghdi [18]. The coupled theory of thermoelasticity can be obtained from Eq. (2.2) in the limiting case $\alpha\to 0$, $\delta=1$ and $t_0\to 0$ as

$$(K_{ij}T_{,i})_{,i}=\rho c_E\frac{\partial T}{\partial t}+\gamma T_0\frac{\partial}{\partial t}(\beta_{ij}u_{i,i}).$$

3 Formulation of the problem

We consider a homogeneous, transversely isotropic, thermally conducting thermoelastic an infinite thick plate with traction free surfaces and thickness $2L$ under hydrostatic initial stress. Let the origin of the coordinate system (x,y,z) is taken at the middle surface of the plate, where $-L\le x\le L$. Since the xy plane is chosen to coincide with the middle surface and y axis normal to it along the thickness, the components of the displacement vector $\mathbf{u}=(u,v,w)$ and temperature T can be written as follows:

$$u=u(x,y,t),\quad v=v(x,y,t),\quad w=0,\quad T=T(x,y,t). \tag{3.1}$$

The reinforcement direction, $\mathbf{b}$ is setting as $\mathbf{b}=(1,0,0)$. According to the above problem formulation, the constitutive relations can be derived from (2.3) as:

$$\sigma_{xx}=(\lambda+2\chi+4\mu_L-2\mu_T+\beta)\frac{\partial u}{\partial x}+(\lambda+\chi)\frac{\partial v}{\partial y}-\beta_{11}(T-T_0), \tag{3.2a}$$

$$\sigma_{yy}=(\lambda+2\mu_T)\frac{\partial v}{\partial y}+(\lambda+\chi)\frac{\partial u}{\partial x}-\beta_{22}(T-T_0), \tag{3.2b}$$

$$\sigma_{xy}=\mu_L\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big). \tag{3.2c}$$

The the forms of the constants β_{11} and β_{22} are

$$\begin{aligned}\beta_{11}&=(2\lambda+3\chi+4\mu_L-2\mu_T+\beta)\alpha_{11}+(\lambda+\alpha)\alpha_{22},\\ \beta_{22}&=(2\lambda+\chi)\alpha_{11}+(\lambda+2\mu_T)\alpha_{22},\end{aligned}$$

where α_{11} and α_{22} are the coefficients of thermal expansion.

In similar way of driving (3.2a)-(3.2c), the equations of motion along x and y directions can be obtained as follows

$$\begin{aligned}&(\lambda+2(\chi+\mu_T)+4(\mu_L-\mu_T)+\beta+\sigma_0)\frac{\partial^2 u}{\partial x^2}+(\sigma_0+\mu_L)\frac{\partial^2 u}{\partial y^2}\\ &\quad+(\chi+\lambda+\mu_L+\mu_L)\frac{\partial^2 v}{\partial x\partial y}-\beta_{11}\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2},\end{aligned} \tag{3.3a}$$

$$\begin{aligned}&(\lambda+2\mu_T+\sigma_0)\frac{\partial^2 v}{\partial y^2}+(\sigma_0+\mu_L)\frac{\partial^2 v}{\partial x^2}+(\chi+\lambda+\mu_L+\mu_L)\frac{\partial^2 u}{\partial x\partial y}\\ &\quad-\beta_{22}\frac{\partial T}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2},\end{aligned} \tag{3.3b}$$

where σ_0 is the initial stress. According to the aforementioned above assumptions, the heat equation can be obtained using (2.2) as follows:

$$K_{11}\frac{\partial^2 T}{\partial x^2}+K_{22}\frac{\partial^2 T}{\partial y^2}=\Big(\delta+t_0\frac{\partial^\alpha}{\partial t^\alpha}\Big)\Big[\rho C_E\frac{\partial T}{\partial t}+T_0\frac{\partial}{\partial t}\Big(\beta_{11}\frac{\partial u}{\partial x}+\beta_{22}\frac{\partial v}{\partial y}\Big)\Big]. \tag{3.4}$$

To simplify the above equations, the following non-dimensional variables are assumed

$$\left.\begin{array}{lll} x'=c_0\eta x, & y'=c_0\eta y, & u'=c_0\eta u, \\ v'=c_0\eta v, & \theta=\beta_{11}(T-T_0)/(\lambda+2\mu_T), & t'=c_0^2\eta t, \\ t_0'=c_0^2\eta t_0, & \sigma_{ij}'=\sigma_{ij}/(\rho c_0^2), & \sigma_0'=\sigma_0/(\rho c_0^2), \end{array}\right\} \tag{3.5}$$

where

$$c_0^2=\frac{A_{11}}{\rho},\quad \eta=\frac{\rho C_E}{K_{11}},\quad A_{11}=\lambda+2(\chi+\mu_T)+4(\mu_L-\mu_T)+\beta.$$

The above governing equations, with the help of Eq. (3.5) may be recast into the dimensionless form after suppressing the primes as:

$$(1+\sigma_0)\frac{\partial^2 u}{\partial x^2}+(\sigma_0+B_4)\frac{\partial^2 u}{\partial y^2}+(B_1+B_4)\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \tag{3.6a}$$

$$(B_2+\sigma_0)\frac{\partial^2 v}{\partial y^2}+(\sigma_0+B_4)\frac{\partial^2 v}{\partial x^2}+(B_1+B_4)\frac{\partial^2 u}{\partial x\partial y}-B_3\frac{\partial\theta}{\partial y}=\frac{\partial^2 v}{\partial t^2}, \tag{3.6b}$$

$$\frac{\partial^2\theta}{\partial x^2}+\varepsilon_1\frac{\partial^2\theta}{\partial y^2}=\Big(\delta+t_0\frac{\partial^\alpha}{\partial t^\alpha}\Big)\Big[\frac{\partial\theta}{\partial t}+\frac{\partial}{\partial t}\Big(\varepsilon_2\frac{\partial u}{\partial x}+\varepsilon_3\frac{\partial v}{\partial y}\Big)\Big], \tag{3.6c}$$

$$\sigma_{xx}=\frac{\partial u}{\partial x}+B_1\frac{\partial v}{\partial y}-\theta, \tag{3.6d}$$

$$\sigma_{yy}=B_1\frac{\partial u}{\partial x}+B_2\frac{\partial v}{\partial y}-B_3\theta, \tag{3.6e}$$

$$\sigma_{xy}=B_4\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big). \tag{3.6f}$$

Where

$$B_1=\frac{A_{12}}{A_{11}},\quad B_2=\frac{A_{22}}{A_{11}},\quad B_3=\frac{\beta_{22}}{\beta_{11}},\quad B_4=\frac{\mu_L}{A_{11}},\quad \varepsilon_1=\frac{K_{22}}{K_{11}},$$

$$A_{11}=\lambda+2(\chi+\mu_T)+4(\mu_L-\mu_T)+\beta,\quad \varepsilon_2=\frac{\beta_{22}\beta_{11}T_0}{\rho C_E A_{11}},$$

$$A_{12}=\chi+\lambda+\mu_L,\quad A_{22}=\lambda+2\mu_T,\quad \varepsilon_2=\frac{\beta_{11}^2 T_0}{\rho C_E A_{11}}.$$

4 Initial and boundary conditions

The above equations are solved subjected to the initial conditions,

$$u(x,y,t)=v(x,y,t)=\theta(x,y,t)=0,\qquad t=0,$$

$$\frac{\partial u(x,y,t)}{\partial t}=\frac{\partial v(x,y,t)}{\partial t}=\frac{\partial\theta(x,y,t)}{\partial t}=0,\qquad t=0.$$

The boundary conditions for the problem may be taken as

$$\sigma_{xx}(L,y,t)=-P,\qquad \sigma_{xy}(L,y,t)=0,\qquad \frac{\partial\theta}{\partial x}(L,y,t)=0, \tag{4.1a}$$

$$u(-L,y,t)=0,\qquad v(-L,y,t)=0,\qquad \theta(-L,y,t)=0. \tag{4.1b}$$

5 Normal mode analysis

The normal mode analysis gives exact solutions without any assumed restrictions on temperature, displacement, and stress distributions. It is applied to a wide range of problems in different branches. It can be applied to boundary-layer problems, which are described by the linearized Navier–Stokes equations in electro hydrodynamics. The normal mode analysis is, in fact, to look for the solution in the Fourier transformed domain, assuming that all the field quantities are sufficiently smooth on the real line so that the normal mode analysis of these functions exists. The normal mode expansion method has been proposed by Cheng and Zhang [37] for modeling the thermoelastic generation process of elastic waveforms in an isotropic plate.

The solution of the considered physical variable can be decomposed in terms of normal modes as the following form

$$[u,v,\theta,\sigma_{ij}](x,y,t)=[u^*,v^*,\theta^*,\sigma_{ij}^*](x)e^{(\omega t+iay)}, \tag{5.1}$$

where ω is the (complex) frequency constant, $i=\sqrt{-1}$, a is the wave number in the y direction, and $u^*(x)$, $v^*(x)$, $\theta^*(x)$, and $\sigma_{ij}^*(x)$ are the amplitudes of the field quantities. Using (5.1), Eqs. (3.6a)-(3.6f) take the forms

$$\Big(\frac{d^2}{dx^2}-g_1\Big)u^*+g_2\frac{dv^*}{dx}=g_3\frac{d\theta^*}{dx}, \tag{5.2a}$$

$$\Big(\frac{d^2}{dx^2}-g_4\Big)v^*+g_5\frac{du^*}{dx}=g_6\theta^*, \tag{5.2b}$$

$$\Big(\frac{d^2}{dx^2}-g_7\Big)\theta^*=g_8\frac{du^*}{dx}+g_9v^*, \tag{5.2c}$$

$$\sigma_{xx}^*=\frac{du^*}{dx}+iaB_1v^*-\theta^*, \tag{5.2d}$$

$$\sigma_{yy}^*=B_1\frac{du^*}{dx}+iaB_2v^*-B_3\theta^*, \tag{5.2e}$$

$$\sigma_{xy}^*=B_4\Big(iau^*+\frac{dv^*}{dx}\Big), \tag{5.2f}$$

where

$$g_1=\left(\frac{a^2(\sigma_0+B_4)}{(1+\sigma_0)}+\frac{\omega^2}{(1+\sigma_0)}\right),\qquad g_2=\left(\frac{ia(B_1+B_4)}{(1+\sigma_0)}\right),\qquad g_3=\frac{1}{(1+\sigma_0)},$$

$$g_4=\left[\frac{a^2(B_2+\sigma_0)}{(\sigma_0+B_4)}+\frac{\omega^2}{(\sigma_0+B_4)}\right],\qquad g_5=\frac{ia(B_1+B_4)}{(\sigma_0+B_4)},\qquad g_6=\frac{iaB_3}{(\sigma_0+B_4)},$$
$$g_7=\left[\omega^2\varepsilon_1+\omega(\delta+t_0\omega^\alpha)\right],\qquad g_8=\omega(1+t_0\omega^\alpha)\varepsilon_2,\qquad g_9=ia\varepsilon_3\omega(1+t_0\omega^\alpha).$$

Eliminating $\theta^*(x)$ and $v^*(x)$ in Eqs. (5.2a)-(5.2c), we obtain

$$(D^6-AD^4+BD^2-C)u^*(x)=0, \tag{5.3}$$

with

$$A=\frac{h_1h_5-h_6g_3h_4-h_3g_2}{h_6-g_8g_2},\qquad B=\frac{h_2h_5-h_1h_6-h_3h_4}{h_5-g_8g_2},\qquad C=\frac{-h_6h_2}{h_5-g_8g_2},$$
$$h_1=g_4+g_7,\qquad h_2=g_4g_7-g_9g_6,\qquad h_3=g_8g_4+g_5g_9,$$
$$h_4=g_9g_3+g_2g_7,\qquad h_5=g_2g_8-g_9,\qquad h_6=g_1g_9.$$

The above equation can be factorized as

$$(D^2-k_1^2)(D^2-k_2^2)(D^2-k_3^2)u^*(x)=0, \tag{5.4}$$

where k_n^2 $(n=1,2,3)$ are the roots of the following characteristic equation

$$k^6-Ak^4+Bk^2-C=0. \tag{5.5}$$

The solution of Eq. (5.4) is given by

$$u^*(x)=\sum_{n=1}^{3}(M_{1n}(a,\omega)e^{-k_nx}+M_{2n}(a,\omega)e^{k_nx}). \tag{5.6}$$

In a similar manner, we get

$$(D^6-AD^4+BD^2-C)\{v(x),\theta^*(x)\}=0. \tag{5.7}$$

Similarly

$$\theta^*(x)=\sum_{n=1}^{3}(M'_{1n}(a,\omega)e^{-k_nx}+M'_{2n}(a,\omega)e^{k_nx}), \tag{5.8a}$$
$$v^*(x)=\sum_{n=1}^{3}(M_{1n}''(a,\omega)e^{-k_nx}+M_{2n}''(a,\omega)e^{k_nx}), \tag{5.8b}$$

where M_n, M'_n, and M_n'' are some parameters depending on a and ω. Substituting Eqs. (5.5)-(5.8b) into Eqs. (5.2a)-(5.2c), we obtain the following relation:

$$M'_{1n}(a,\omega)=H_{1n}M_{1n}(a,\omega),\quad M'_{2n}(a,\omega)=H_{2n}M_{2n}(a,\omega),$$
$$M_n''(a,\omega)=H_{2n}M_n(a,\omega),\quad n=1,2,3,$$

where

$$H_{1n}=\frac{-(g_8k_n^3-h_3k_n)}{k_n^4-h_1k_n^2+h_2}, \qquad H_{2n}=-H_{1n},$$
$$H_{3n}=\frac{1}{g_9}[(k_n^2-g_7)H_{1n}+g_8k_n], \qquad H_{4n}=\frac{1}{g_9}[(k_n^2-g_7)H_{1n}-g_8k_n].$$

Thus, we have

$$\theta^*(x)=\sum_{n=1}^{3}(H_{1n}M_{1n}(a,\omega)e^{-k_nx}+H_{2n}M_{2n}(a,\omega)e^{k_nx}), \tag{5.9a}$$
$$v^*(x)=\sum_{n=1}^{3}(H_{3n}M_{1n}(a,\omega)e^{-k_nx}+H_{4n}M_{2n}(a,\omega)e^{k_nx}). \tag{5.9b}$$

Substituting Eqs. (5.5), (5.9a), and (5.9b) into Eqs. (5.2d)-(5.2f), we obtain

$$\sigma^*_{xx}(x)=\sum_{n=1}^{3}(H_{5n}M_{1n}(a,\omega)e^{-k_nx}+H_{6n}M_{2n}(a,\omega)e^{k_nx}), \tag{5.10a}$$
$$\sigma^*_{yy}(x)=\sum_{n=1}^{3}(H_{7n}M_{1n}(a,\omega)e^{-k_nx}+H_{8n}M_{2n}(a,\omega)e^{k_nx}), \tag{5.10b}$$
$$\sigma^*_{xy}(x)=\sum_{n=1}^{3}(H_{9n}M_{1n}(a,\omega)e^{-k_nx}+H_{10n}M_{2n}(a,\omega)e^{k_nx}), \tag{5.10c}$$

where

$$H_{5n}=-k_n+iaB_1H_{3n}-H_{1n}, \qquad H_{6n}=k_n+iaB_1H_{4n}-H_{2n},$$
$$H_{7n}=-B_1k_n+iaB_2H_{3n}-B_3H_{1n}, \qquad H_{10n}=iaB_4+B_4k_nH_{4n},$$
$$H_{9n}=iaB_4-B_4k_nH_{3n}, \qquad H_{8n}=B_1k_n+iaB_2H_{4n}-B_3H_{2n}.$$

Substituting the expressions of the variables considered into the boundary conditions, we obtain

$$\sigma^*_{xx}|_{x=L}=\sum_{n=1}^{3}(H_{5n}M_{1n}(a,\omega)e^{-k_nL}+H_{6n}M_{2n}(a,\omega)e^{k_nL})=-P^*, \tag{5.11a}$$
$$\sigma^*_{xy}|_{x=L}=\sum_{n=1}^{3}(H_{9n}M_{1n}(a,\omega)e^{-k_nL}+H_{10n}M_{2n}(a,\omega)e^{k_nL})=0, \tag{5.11b}$$
$$\frac{\partial\theta^*}{\partial x}\Big|_{x=L}=\sum_{n=1}^{3}(-k_nH_{1n}M_{1n}(a,\omega)e^{-k_nL}+k_nH_{2n}M_{2n}(a,\omega)e^{k_nL})=0, \tag{5.11c}$$
$$u^*|_{x=-L}=\sum_{n=1}^{3}(M_{1n}(a,\omega)e^{k_nL}+M_{2n}(a,\omega)e^{-k_nL})=0, \tag{5.11d}$$

$$\theta^*|_{x=-L}=\sum_{n=1}^{3}(H_{1n}M_{1n}(a,\omega)e^{k_nL}+H_{2n}M_{2n}(a,\omega)e^{-k_nL})=0, \tag{5.11e}$$

$$v^*|_{x=-L}=\sum_{n=1}^{3}(H_{3n}M_{1n}(a,\omega)e^{k_nL}+H_{3n}M_{2n}(a,\omega)e^{-k_nL})=0. \tag{5.11f}$$

After applying the inverse of matrix method, we have the values of the three constants M_{1j} and M_{2j} $(j=1,2,3)$

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{21} & M_{22} & M_{23} \end{pmatrix}^{Tr}=E^{-1}\times F, \tag{5.12}$$

where

$$F=\begin{pmatrix} -P^* & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{Tr},$$

$$E=\begin{pmatrix} H_{51}e^{-k_1L} & H_{52}e^{-k_2L} & H_{53}e^{-k_3L} & H_{61}e^{k_1L} & H_{62}e^{k_2L} & H_{63}e^{k_3L} \\ H_{91}e^{-k_1L} & H_{92}e^{-k_2L} & H_{93}e^{-k_3L} & H_{101}e^{k_1L} & H_{102}e^{k_2L} & H_{103}e^{k_3L} \\ -k_1H_{11}e^{-k_1L} & -k_2H_{12}e^{-k_2L} & -k_3H_{13}e^{-k_3L} & k_1H_{21}e^{k_1L} & k_2H_{22}e^{k_2L} & k_3H_{23}e^{k_3L} \\ e^{k_1L} & e^{k_2L} & e^{k_3L} & e^{-k_1L} & e^{-k_2L} & e^{-k_3L} \\ H_{11}e^{k_1L} & H_{12}e^{k_2L} & H_{13}e^{k_3L} & H_{21}e^{-k_1L} & H_{22}e^{-k_2L} & H_{23}e^{-k_3L} \\ H_{31}e^{k_1L} & H_{32}e^{k_2L} & H_{33}e^{k_3L} & H_{41}e^{-k_1L} & H_{42}e^{-k_2L} & H_{43}e^{-k_3L} \end{pmatrix}.$$

Hence, we obtain the expressions for the displacements, the temperature distribution, and another physical quantities of the plate muscles.

6 Particular and special cases

6.1 Equation of coupled thermoelasticity

The equations of the coupled thermoelasticity (**CTE** theory) are obtained when $t_0=0$ and $\delta=1$.

6.2 Equations of generalized thermoelasticity with one relaxation time

The equations of the Lord-Shulman (**LS** theory), are retrieved when $\alpha\to 1$, $t_0>0$ and $\delta=1$.

6.3 Equations of generalized thermoelasticity without energy dissipation

The equations of the generalized thermoelasticity without energy dissipation (the linearized **GN** theory of type II) are obtained when $\alpha\to 1$, $\delta=0$ and $t_0=1$.

Also all the results reduce to the classical isotropic results when the anisotropic parameters for the fibre-reinforced medium tend to zero (if necessary writing $\chi=0$, $\beta=0$ and considering $|\mu_L-\mu_T|\to 0$.

7 Numerical results

We now consider a numerical example for which computational results are given. The results depict the variations of normal displacement, normal force stress and temperature distribution in the context of fractional order thermoelasticity theory. To study the effect of reinforcement on wave propagation, we use the following physical constants for generalized fibre-reinforced thermoelastic materials:

$$\begin{aligned}
&\lambda=5.65\times10^{10}\mathrm{N/m^2}, \quad \mu_T=2.46\times10^{10}\mathrm{N/m^2}, \quad \mu_L=5.66\times10^{10}\mathrm{N/m^2},\\
&\chi=-1.28\times10^{10}\mathrm{N/m^2}, \quad \beta=220.9\times10^{10}\mathrm{N/m^2}, \quad C_E=0.787\times10^{3}\mathrm{J/(kgK)},\\
&\rho=2660\mathrm{kg/m^3}, \quad \alpha_{11}=0.017\times10^{-4}\mathrm{K^{-1}}, \quad \alpha_{22}=0.015\times10^{-4}\mathrm{K^{-1}},\\
&K_{11}=0.0921\times10^{3}\mathrm{J\cdot m^{-1}\cdot s^{-1}\cdot K^{-1}}, \quad K_{22}=0.0963\times10^{3}\mathrm{J\cdot\, m^{-1}\cdot\, s^{-1}\cdot K^{-1}},\\
&\tau_0=0.02, \quad a=1, \quad T_0=293\mathrm{K}, \quad L=1,\\
&P^*=0.5, \quad \omega=\omega_0+i\xi, \quad \omega_0=2, \quad \xi=1.
\end{aligned}$$

The calculations are carried out for a time of $t=0.3$. The numerical technique outlined above is used for the distribution of the real part of the thermal temperature θ, the displacements u and v, the distributions of stresses σ_{xx}, σ_{yy}, and σ_{xy} for the problem. The field quantities including temperature, displacement components u, v, and stress componentsσ_{xx}, σ_{yy}, and σ_{xy} depend not only on space x and time t, but also on fractional order α. Here all the variables are taken in the non-dimensional forms.

Figs. 1-6 are drawn to give comparison of the results obtained for displacements, temperature and stresses for three models of Lord and Shulman model, fractional order model and classical model, against the thickness x for different values of α at $y=1$. Three different values of α ($\alpha=1$,0.75 and $\alpha=0.25$ i.e., in the absence and the presence of the fractional) are considered. It can be found from Figs. 1-6 that, the physical quantities

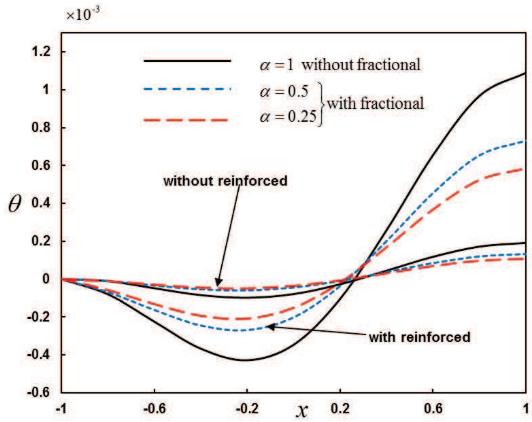

Figure 1: Dependence of temperature θ on distance for different values of fractional order parameter α fractional in the absence and presence of fiber-reinforcement.

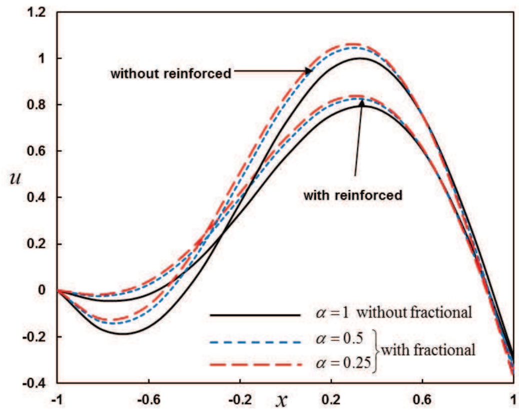

Figure 2: Dependence of horizontal displacement distribution u on distance for different values of fractional order parameter α fractional in the absence and presence of fiber-reinforcement.

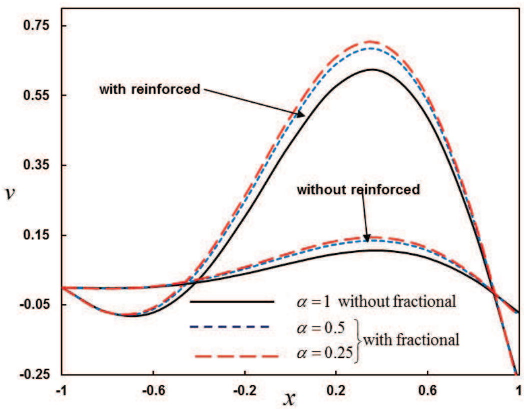

Figure 3: Dependence of vertical displacement v on distance for different values of fractional order parameter α fractional in the absence and presence of fiber-reinforcement.

depend not only on time and space, but also depend on the characteristic parameter of fractional order thermoelasticity theory. We notice that the results for the temperature, the displacement and stress distribution with the fractional order parameter included in the heat equation are distinctly different from those without the fractional order parameter in the heat equation. The different values of the parameter α with wide range $(0<\alpha\leq 1)$ cover the two cases of the conductivity; $(0<\alpha<1)$ for weak conductivity and $\alpha=1$ for normal conductivity (ordinary heat conduction equation).

Fig. 1 shows the variation of temperature with x and it indicates that temperature field has maximum value at the boundary $x=L=1$ and then decreases to zero at $x=-L=-1$. The $y=0$ represents the plane of the crack, which is symmetric with respect to the y plane. The effect of fractional order parameter on temperature increases the value of the real part of of θ. As shown in Fig. 2, horizontal displacement u increases near the

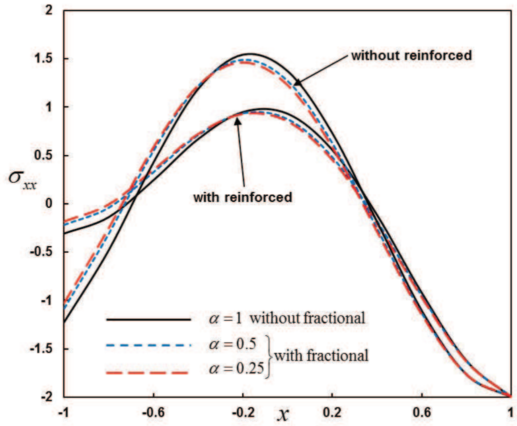

Figure 4: Dependence of stress σ_{xx} on distance for different values of fractional order parameter α fractional in the absence and presence of fiber-reinforcement.

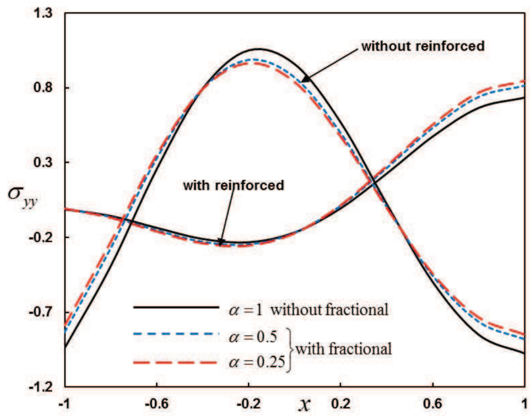

Figure 5: Dependence of stress σ_{yy} on distance for different values of fractional order parameter α fractional in the absence and presence of fiber-reinforcement.

boundary $x=L=1$, then smooth decreases again to reach its minimum magnitude just at about the plate end $x=-L=-1$. The values of u for $\alpha=0.25,0.75$ are larger compared to those for $\alpha=1$. Fig. 3 shows vertical displacement v. We can see that displacement component v always starts from a negative value and terminates at the zero value. The values of v for $\alpha=0.25,0.75$ are larger compared to those for $\alpha=1$.

It is observed from these figures that on the rigid base at $x=-L$, the temperature and displacements are zero which confirms the assumed boundary conditions. On the upper surface of the plate, $x=L$ is assumed to be thermally insulated and the displacements are maxima which supports the physical fact.

The stress component σ_{xx} reaches coincidence with a negative value (Fig. 4). The behaviors of the two cases with different values of fractional are similar. The fractional

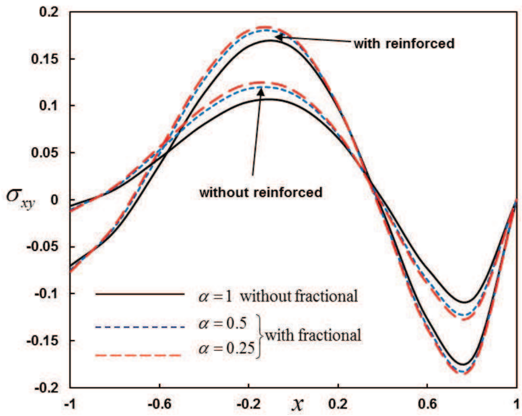

Figure 6: Dependence of stress σ_{xy} on distance for different values of fractional order parameter α fractional in the absence and presence of fiber-reinforcement.

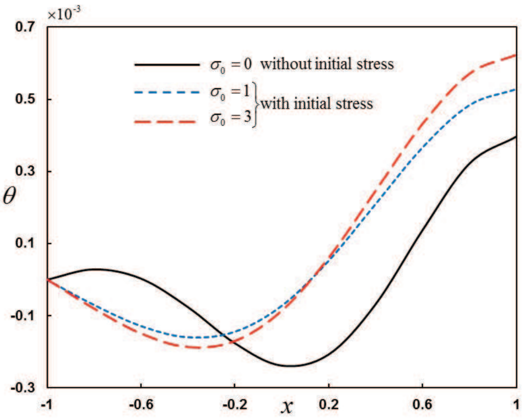

Figure 7: The temperature distribution in the case of material with fractional in the absence and presence of initial stress.

order decreases the amplitudes of the stress. Figs. 5 and 6 show the same behavior as that found in Fig. 4. Fig. 6 shows that stress component σ_{xy} satisfies the boundary condition at $x=L$ and has a different behavior compared to that of σ_{yy}. These trends obey elastic and thermoelastic properties of the solid under investigation.

Figs. 1-6 show also the variation of the physical quantities with space x at $t=0.15$ under two types with reinforcement and without reinforcement (i.e., $\alpha=0$, $\beta=0$ and $\mu_L-\mu_T=0$). The values of u, σ_{xx}, and σ_{yy} are evidently smaller with reinforcement when compared to those in the absence of reinforcement. The values of thermal stress σ_{xy} and displacement v and the values of θ are evidently larger with reinforcement when compared to those in the absence of reinforcement. It is clear from the above investigation that the surface waves in the fibre-reinforced medium are affected by the reinforced parameters.

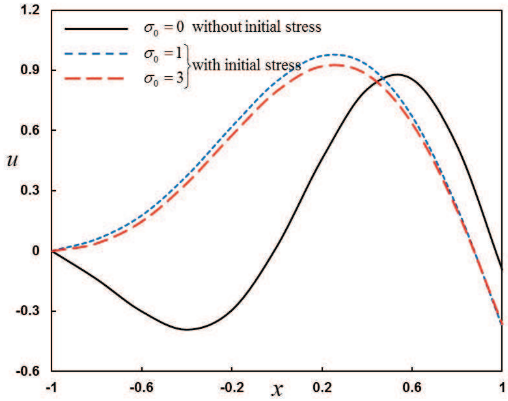

Figure 8: The displacement u distribution in the case of material with fractional in the absence and presence of initial stress.

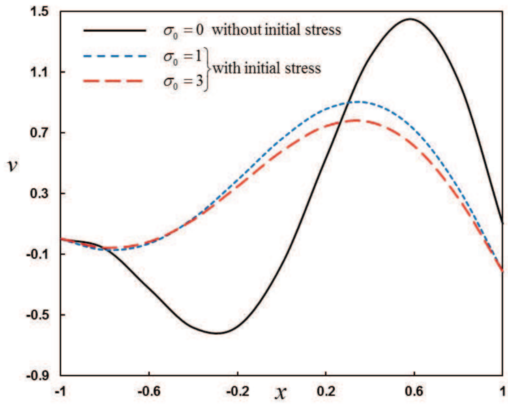

Figure 9: The displacement v distribution in the case of material with fractional in the absence and presence of initial stress.

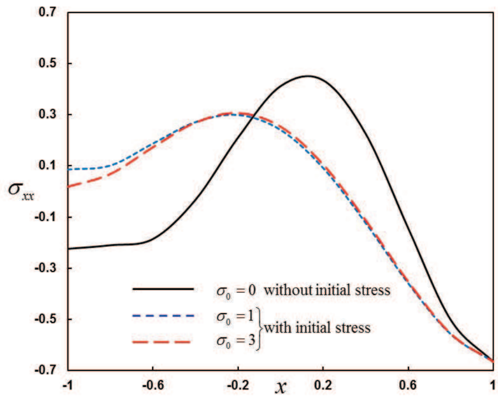

Figure 10: The stress σ_{xx} distribution in the case of material with fractional in the absence and presence of initial stress.

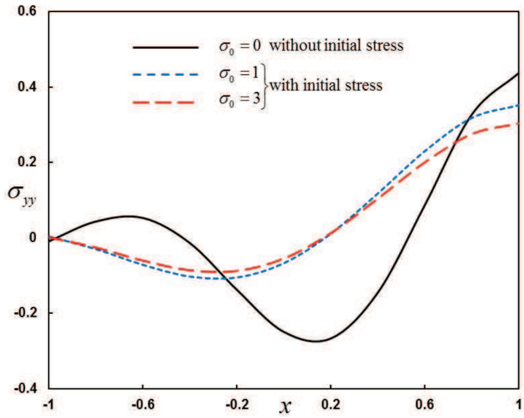

Figure 11: The stress σ_{yy} distribution in the case of material with fractional in the absence and presence of initial stress.

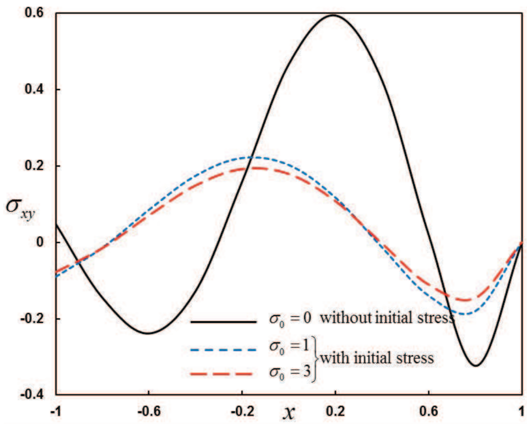

Figure 12: The stress σ_{xy} distribution in the case of material with fractional in the absence and presence of initial stress.

Figs. 7-12 exhibit the variation of the temperature, displacement components u, v, and stress components σ_{xx}, σ_{yy}, and σ_{xy} with distance x under the fractional order theory theory at y=1 for three different values of initial stress (σ_0=0,1,3) at α=0.5. It is clear from the figures that the surface waves in the fibre-reinforced medium are depend not only on the state and space variables t, x, and y, but also on the initial stress σ_0. It has been observed that the initial stress σ_0 plays a vital role on the development of temperature, stress, and displacement fields.

8 Conclusions

Analytical solutions based on the normal mode analysis for the themoelastic problem in solids have been developed and utilized. The stress distributions and the temperature are evaluated as functions of the distance. The effects of anisotropy, hydrostatic initial stress and fractional are studied on all the quantities. Analytical solutions based upon normal mode analysis for thermoelasticity with fractional order in solids have been developed and utilized. The computations have revealed that:

1. The presence of fractional order parameter α plays a significant role in all the physical quantities. Therefore, the presence of fractional order parameter in the current model is of significance.
2. The fibre-reinforcement has an important role on the distributions of the field quantities.
3. The method which used in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.
4. The theories of coupled thermoelasticity (**CTE**), and generalized thermoelasticity with one relaxation time (**LS**) and without energy dissipation (**GN**) can be obtained as limited cases.
5. Deformation of a body depends on the nature of the forces applied as well as the type of boundary conditions.
6. The variations of all the quantities show appreciable effect with and without dependence of initial stress.
7. The method, which is used in the present article, is applicable to a wide range of problems in thermodynamics and thermoelasticity.
8. According to the numerical results and its graphs, a conclusion about the new theory of thermoelasticity has been constructed. The result provides a motivation to investigate conducting materials as a new class of applicable materials.

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