Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 1, pp. 1-20

DOI: 10.4208/aamm.10-m1055
February 2012

Convergence Analysis of the Spectral Methods for Weakly Singular Volterra Integro-Differential Equations with Smooth Solutions

Yunxia Wei[1] and Yanping Chen[2,*]

[1] *School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou 510275, China*
[2] *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China*

Received 16 September 2010; Accepted (in revised version) 16 September 2011

Available online 13 December 2011

Abstract. The theory of a class of spectral methods is extended to Volterra integro-differential equations which contain a weakly singular kernel $(t-s)^{-\mu}$ with $0 < \mu < 1$. In this work, we consider the case when the underlying solutions of weakly singular Volterra integro-differential equations are sufficiently smooth. We provide a rigorous error analysis for the spectral methods, which shows that both the errors of approximate solutions and the errors of approximate derivatives of the solutions decay exponentially in L^∞-norm and weighted L^2-norm. The numerical examples are given to illustrate the theoretical results.

AMS subject classifications: 45J05, 65R20

Key words: Volterra integro-differential equations, weakly singular kernels, spectral methods, convergence analysis.

1 Introduction

Volterra integro-differential equations (VIDEs) arise widely in mathematical models of certain biological and physical phenomena. Due to the wide application of these equations, they must be solved successfully with efficient numerical methods. Piecewise polynomial collocation methods have been introduced in [7]. In [27], Tang discussed the application of a class of spline collocation methods to weakly singular Volterra integro-differential equations. Polynomial spline collocation methods were investigated in [5,24,29]. Bologna [4] found an asymptotic solution for first and second order

*Corresponding author.
URL: `http://202.116.32.252/user_info.asp?usernamesp=%B3%C2%D1%DE%C6%BC`
Email: yunxiawei@126.com (Y. X. Wei), yanpingchen@scnu.edu.cn (Y. P. Chen)

VIDEs containing an arbitrary kernel. In [32], sinc-collocation method was developed to approximate the second order VIDEs with boundary conditions.

So far, very few works have touched the spectral approximations to weakly singular VIDEs. Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve certain partial differential equations [14, 20, 25, 31]. In practice, spectral methods have excellent error properties with the so-called "exponential convergence" being the fastest possible. Recently, the present authors have developed the spectral methods for the solutions of Volterra integral equations (VIEs) of the second kind [10,11,28], pantograph-type delay differential equations [1, 2] and singularly perturbed problems [18]. Moreover, in [30], we apply the Legendre spectral collocation methods to approximate the solutions of second order VIDEs. The main purpose of this work is to provide the Jacobi spectral collocation methods for weakly singular VIDEs. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence.

The Volterra integro-differential equation that we shall study in details reads:

$$y'(t) = a(t)y(t) + b(t) + (\nu_\mu y)(t), \quad t \in I := [0, T], \quad y(0) = y_0, \tag{1.1}$$

where $\nu_\mu : C(I) \to C(I)$ is defined by

$$(\nu_\mu \phi)(t) := \int_0^t (t-s)^{-\mu} K(t,s)\phi(s)ds,$$

with $0 < \mu < 1$, the functions $a(t)$, $b(t) \in C(I)$, $y(t)$ is the unknown function and $K \in C(I \times I)$, $K(t,t) \neq 0$ for $t \in I$. Equations of this type arise as model equations for describing turbulent diffusion problems. The numerical treatment of the Volterra integro-differential equation (1.1) is not simple, mainly due to the fact that the solutions of (1.1) usually have a weak singularity at $t = 0$. As discussed in [6], the second derivative of the solution $y(t)$ behaves like

$$y''(t) \sim t^{-\mu}.$$

We point out that for (1.1) without the singular kernel (i.e., $\mu = 0$) spectral methods and the corresponding error analysis have been provided recently [16]; see also [28] and [1] for spectral methods to Volterra integral equations and pantograph-type delay differential equations. In both cases, the underlying solutions are smooth.

In this work, we will consider a special case, namely, the exact solutions of (1.1) are smooth (see also [8]). In this case, the Jacobi spectral collocation method can be applied directly. The organization of this paper is as follows: the spectral approaches for the VIDEs with weakly singular kernels are presented in Section 2, and some lemmas useful for establishing the convergence results are given in Section 3. In Section 4 the convergence analysis is outlined, and Section 5 contains numerical results, which will be used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with conclusions and future work.