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Mathematical Simulation of Cloaking Metamaterial Structures

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Abstract. In this paper we present a rigorous derivation of the material parameters for both the cylinder and rectangle cloaking structures. Numerical results using these material parameters are presented to demonstrate the cloaking effect.

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Key words: Maxwell's equations, metamaterial, finite element method, cloaking structures.

1 Introduction

In recent years, inspired by the pioneering work of Pendry et al. [15] and Leonhardt [10], there are lots of work devoted to the study of using metamaterials (e.g., [5,7,11]) to construct invisibility cloaks of different shapes (e.g., [1,9,14,18–20]). More details and references on cloaking can be found in recent reviews [2,6]. The basic principal behind this is the so-called transformation optics [10,15], which uses the coordinate transformation to design the material parameters to steer the light around some regions. Unfortunately, very few papers provided a clear derivation of the material parameters so that many researchers wasted a great deal of time on guessing those parameters and still could not obtain nice cloak results.

The main goal of this paper is to present a rigorous derivation of the material parameters for both the cylinder and rectangle cloaking structures. Detailed numerical results are provided to demonstrate our correct derivation and the cloaking effect achieved using these material parameters. Hopefully these will serve as benchmark problems so that other researchers can easily reproduce these models and inspire further advance in this area.

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2 The mathematical formulation

Modeling of electromagnetic phenomena at a fixed frequency ω is governed by the full Maxwell's equations (assuming a time harmonic variation of $\exp(i\omega t)$):

$$\nabla \times \boldsymbol{E} + i\omega\mu\boldsymbol{H} = 0, \qquad \nabla \times \boldsymbol{H} - i\omega\epsilon\boldsymbol{E} = 0, \tag{2.1}$$

where $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$ are the electric and magnetic fields, ϵ and μ are the permittivity and permeability of the material.

A very important property for Maxwell's equations is that Maxwell's equations are form invariant under coordinate transformations [16]. More specifically, under a coordinate transformation $\boldsymbol{x}' = \boldsymbol{x}'(\boldsymbol{x})$, the Eq. (2.1) keeps the same form in the transformed coordinate system [14]:

$$\nabla' \times \boldsymbol{E}' + i\omega\mu'\boldsymbol{H}' = 0, \qquad \nabla' \times \boldsymbol{H}' - i\omega\epsilon'\boldsymbol{E}' = 0, \tag{2.2}$$

where all new variables are given by

$$\boldsymbol{E}'(\boldsymbol{x}') = A^{-T}\boldsymbol{E}(\boldsymbol{x}), \quad \boldsymbol{H}'(\boldsymbol{x}') = A^{-T}\boldsymbol{H}(\boldsymbol{x}), \quad A = (A_{ij}), \quad A_{ij} = \frac{\partial x_i'}{\partial x_j}, \tag{2.3}$$

and

$$\mu'(\boldsymbol{x}') = \frac{A\mu(\boldsymbol{x})A^T}{\det(A)}, \qquad \epsilon'(\boldsymbol{x}') = \frac{A\epsilon(\boldsymbol{x})A^T}{\det(A)}. \tag{2.4}$$

2.1 Cylindrical cloak

Following [15], cloaking a central cylindrical region R_1 by a concentric cylindrical region of radius R_2 can be done using the following coordinate transformation:

$$r'(r,\theta) = \frac{R_2 - R_1}{R_2} r + R_1, \tag{2.5a}$$

$$\theta'(r,\theta) = \theta. \tag{2.5b}$$

Since the COMSOL solver is based on Cartesian coordinates, we have to transform the material parameters given in polar coordinates to Cartesian coordinates. In polar coordinates, we have

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \tan^{-1}\frac{x_2}{x_1}, \tag{2.6}$$

where we use the traditional notation: a point (x_1, x_2) in Cartesian coordinate system corresponds to a point (r,θ) in polar coordinate system.

From (2.6) and the relation $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we can obtain

$$\frac{\partial r}{\partial x_1} = \frac{x_1}{r} = \cos\theta, \qquad \frac{\partial r}{\partial x_2} = \frac{x_2}{r} = \sin\theta, \tag{2.7a}$$

$$\frac{\partial \theta}{\partial x_1} = -\frac{x_2}{r^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial x_2} = \frac{x_1}{r^2} = \frac{\cos\theta}{r}. \tag{2.7b}$$