Convergence Analysis of the Spectral Methods for Weakly Singular Volterra Integro-Differential Equations with Smooth Solutions

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Abstract. The theory of a class of spectral methods is extended to Volterra integro-differential equations which contain a weakly singular kernel $(t-s)^{-\mu}$ with $0 < \mu < 1$. In this work, we consider the case when the underlying solutions of weakly singular Volterra integro-differential equations are sufficiently smooth. We provide a rigorous error analysis for the spectral methods, which shows that both the errors of approximate solutions and the errors of approximate derivatives of the solutions decay exponentially in L^{∞}-norm and weighted L^2-norm. The numerical examples are given to illustrate the theoretical results.

1   Introduction

Volterra integro-differential equations (VIDEs) arise widely in mathematical models of certain biological and physical phenomena. Due to the wide application of these equations, they must be solved successfully with efficient numerical methods. Piecewise polynomial collocation methods have been introduced in [7]. In [27], Tang discussed the application of a class of spline collocation methods to weakly singular Volterra integro-differential equations. Polynomial spline collocation methods were investigated in [5,24,29]. Bologna [4] found an asymptotic solution for first and second order

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VIDEs containing an arbitrary kernel. In [32], sinc-collocation method was developed to approximate the second order VIDEs with boundary conditions.

So far, very few works have touched the spectral approximations to weakly singular VIDEs. Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve certain partial differential equations [14, 20, 25, 31]. In practice, spectral methods have excellent error properties with the so-called "exponential convergence" being the fastest possible. Recently, the present authors have developed the spectral methods for the solutions of Volterra integral equations (VIEs) of the second kind [10,11,28], pantograph-type delay differential equations [1,2] and singularly perturbed problems [18]. Moreover, in [30], we apply the Legendre spectral collocation methods to approximate the solutions of second order VIDEs. The main purpose of this work is to provide the Jacobi spectral collocation methods for weakly singular VIDEs. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence.

The Volterra integro-differential equation that we shall study in details reads:

$$y'(t) = a(t)y(t) + b(t) + (v_\mu y)(t), \quad t \in I := [0, T], \quad y(0) = y_0, \tag{1.1}$$

where $v_\mu : C(I) \to C(I)$ is defined by

$$(v_\mu \phi)(t) := \int_0^t (t - s)^{-\mu} K(t, s) \phi(s) ds,$$

with $0 < \mu < 1$, the functions $a(t), b(t) \in C(I)$, $y(t)$ is the unknown function and $K \in C(I \times I)$, $K(t, t) \neq 0$ for $t \in I$. Equations of this type arise as model equations for describing turbulent diffusion problems. The numerical treatment of the Volterra integro-differential equation (1.1) is not simple, mainly due to the fact that the solutions of (1.1) usually have a weak singularity at $t = 0$. As discussed in [6], the second derivative of the solution $y(t)$ behaves like

$$y''(t) \sim t^{-\mu}.$$

We point out that for (1.1) without the singular kernel (i.e., $\mu = 0$) spectral methods and the corresponding error analysis have been provided recently [16]; see also [28] and [1] for spectral methods to Volterra integral equations and pantograph-type delay differential equations. In both cases, the underlying solutions are smooth.

In this work, we will consider a special case, namely, the exact solutions of (1.1) are smooth (see also [8]). In this case, the Jacobi spectral collocation method can be applied directly. The organization of this paper is as follows: the spectral approaches for the VIDEs with weakly singular kernels are presented in Section 2, and some lemmas useful for establishing the convergence results are given in Section 3. In Section 4 the convergence analysis is outlined, and Section 5 contains numerical results, which will be used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with conclusions and future work.

2 Jacobi collocation methods

2.1 Preliminaries

We present here some preliminary materials (see also [3, 9, 26]) which will be used throughout the paper. We first introduce the Legendre polynomials $L_k(x)$, $k = 0, 1, \cdots$, which are the eigenfunctions of the singular Sturm-Liouville problem

$$\left((1 - x^2)L_k'(x)\right)' + k(k+1)L_k(x) = 0.$$

They are mutually orthogonal over the interval $(-1, 1)$:

$$\int_{-1}^{1} L_k(x)L_m(x)dx = 0, \quad \text{whenever } m \neq k.$$

The classical Weierstrass theorem implies that such a system is complete in the space $L^2(-1, 1)$. The associated inner product is

$$(u, v) = \int_{-1}^{1} u(x)v(x)dx.$$

For a given $N \geq 0$, we denote by $\{\tilde{\theta}_k\}_{k=0}^{N}$ the Legendre Gauss points, and by $\{\tilde{\omega}_k\}_{k=0}^{N}$ the corresponding Legendre weights. Then, the Legendre Gauss integration formula is

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{N} f(\tilde{\theta}_k)\tilde{\omega}_k. \tag{2.1}$$

Furthermore, we introduce the Jacobi polynomials (see also [13, 15]) $J_k^{\alpha,\beta}(x)$ of indices $\alpha, \beta > -1$ which are the solutions to singular Sturm-Liouville problems

$$\left((1 - x)^{1+\alpha}(1 + x)^{1+\beta}J_k'(x)\right)' + k(k + \alpha + \beta + 1)(1 - x)^{\alpha}(1 + x)^{\beta}J_k(x) = 0.$$

Hereafter, we denote the Jacobi weight function of index (α, β) by

$$\omega^{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}.$$

We define the "usual" weighted Sobolev spaces:

$$L_{\omega^{\alpha,\beta}}^2(-1, 1) = \left\{u : \int_{-1}^{1} |u(x)|^2 \omega^{\alpha,\beta}(x)dx < +\infty\right\}, \tag{2.2a}$$

$$H_{\omega^{\alpha,\beta}}^l(-1, 1) = \left\{u \in L_{\omega^{\alpha,\beta}}^2(-1, 1) : \partial_x u, \cdots, \partial_x^l u \in L_{\omega^{\alpha,\beta}}^2(-1, 1)\right\}. \tag{2.2b}$$

The space $L_{\omega^{\alpha,\beta}}^2(-1, 1)$ is a Hilbert space for the inner product

$$(u, v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx,$$

and the norm

$$\|u\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)} = \left(\int_{-1}^{1} |u(x)|^2 \omega^{\alpha,\beta}(x)dx \right)^{\frac{1}{2}}.$$

For a given $N \geq 0$, we denote by $\{\hat{\theta}_k\}_{k=0}^{N}$ the Jacobi Gauss points, and by $\{\hat{\omega}_k\}_{k=0}^{N}$ the corresponding Jacobi weights. Then, the Jacobi Gauss integration formula is

$$\int_{-1}^{1} f(x)\omega^{\alpha,\beta}(x)dx \approx \sum_{k=0}^{N} f(\hat{\theta}_k)\hat{\omega}_k. \tag{2.3}$$

In particular, we denote

$$\int_{-1}^{1} f(x)\omega^{-\mu,0}(x)dx \approx \sum_{k=0}^{N} f(\theta_k)\omega_k, \tag{2.4a}$$

$$\int_{-1}^{1} f(x)\omega^{-\mu,-\mu}(x)dx \approx \sum_{k=0}^{N} f(x_k)\nu_k. \tag{2.4b}$$

2.2 Numerical schemes

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t = \frac{T}{2}(1+x),$$

to rewrite (1.1) as follows

$$u'(x) = \tilde{a}(x)u(x) + \tilde{b}(x) + \int_{0}^{\frac{T}{2}(1+x)} \left(\frac{T}{2}(1+x) - s \right)^{-\mu} \bar{K}(x,s)y(s)ds, \tag{2.5}$$

where

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad\qquad \tilde{a}(x) = \frac{T}{2}a\left(\frac{T}{2}(1+x)\right),$$

$$\tilde{b}(x) = \frac{T}{2}b\left(\frac{T}{2}(1+x)\right), \qquad\qquad \bar{K}(x,s) = \frac{T}{2}K\left(\frac{T}{2}(1+x),s\right).$$

Furthermore, to transfer the integral interval $[0, T(1+x)/2]$ to the interval $[-1, x]$, we make a linear transformation: $s = T(1+\tau)/2, \tau \in [-1, x]$. Then Eq. (2.5) becomes

$$u'(x) = \tilde{a}(x)u(x) + \tilde{b}(x) + \int_{-1}^{x} (x-\tau)^{-\mu}\tilde{K}(x,\tau)u(\tau)d\tau, \quad x \in [-1,1], \tag{2.6}$$

with the initial condition

$$u(-1) = u_{-1} = y_0, \tag{2.7}$$

where

$$\tilde{K}(x,\tau) = \left(\frac{T}{2}\right)^{1-\mu}\bar{K}\left(x,\frac{T}{2}(1+\tau)\right).$$

In order that the Jacobi collocation methods are carried out naturally, we restate (2.7) as

$$u(x) = u_{-1} + \int_{-1}^{x} u'(\tau)d\tau. \tag{2.8}$$

Firstly, Eqs. (2.6) and (2.8) hold at the collocation points $\{x_i\}_{i=0}^{N}$ on $[-1,1]$, associated with $\omega^{-\mu,-\mu}$, i.e.,

$$u'(x_i) = \tilde{a}(x_i)u(x_i) + \tilde{b}(x_i) + \int_{-1}^{x_i} (x_i - \tau)^{-\mu}\tilde{K}(x_i,\tau)u(\tau)d\tau, \tag{2.9a}$$

$$u(x_i) = u_{-1} + \int_{-1}^{x_i} u'(\tau)d\tau. \tag{2.9b}$$

In order to obtain high order accuracy for the problem (2.9a)-(2.9b), the main difficulty is to compute the integral term. In particular, for small values of x_i, there is little information available for $u(\tau)$. To overcome this difficulty, we transfer the integral interval $[-1, x_i]$ to a fixed interval $[-1,1]$

$$u'(x_i) = \tilde{a}(x_i)u(x_i) + \tilde{b}(x_i)$$
$$+ \left(\frac{1+x_i}{2}\right)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu}\tilde{K}(x_i,\tau(x_i,\theta))u(\tau(x_i,\theta))d\theta, \tag{2.10a}$$

$$u(x_i) = u_{-1} + \frac{1+x_i}{2} \int_{-1}^{1} u'(\tau(x_i,\theta))d\theta, \tag{2.10b}$$

by using the following variable change

$$\tau = \tau(x_i,\theta) = \frac{1+x_i}{2}\theta + \frac{x_i - 1}{2}, \quad \theta \in [-1,1]. \tag{2.11}$$

Next, using Jacobi Gauss integration formula, the integration term in (2.10a) can be approximated by

$$\int_{-1}^{1} (1-\theta)^{-\mu}\tilde{K}(x_i,\tau(x_i,\theta))u(\tau(x_i,\theta))d\theta \approx \sum_{k=0}^{N} \tilde{K}(x_i,\tau(x_i,\theta_k))u(\tau(x_i,\theta_k))\omega_k, \tag{2.12}$$

where the set $\{\theta_k\}_{k=0}^{N}$ is associated with $\omega^{-\mu,0}$. Similarly, the integration term in (2.10b) can be approximated by

$$\int_{-1}^{1} u'(\tau(x_i,\theta))d\theta \approx \sum_{k=0}^{N} u'(\tau(x_i,\tilde{\theta}_k))\tilde{\omega}_k. \tag{2.13}$$

We use u'_i, $u_i (0 \leq i \leq N)$ to approximate the function value $u'(x_i)$, $u(x_i)(0 \leq i \leq N)$ respectively, and use

$$\tilde{u}'_N(x) = \sum_{j=0}^{N} u'_j F_j(x), \qquad \tilde{u}_N(x) = \sum_{j=0}^{N} u_j F_j(x), \tag{2.14}$$

(although $\tilde{u}_N'(x)$ differs from the exact derivative of $\tilde{u}_N(x)$, we still use this notation) to approximate the function $u'(x)$, $u(x)$, namely,

$$u'(x_i) \approx u_i', \qquad u(x_i) \approx u_i, \qquad u'(x) \approx \tilde{u}_N'(x), \qquad u(x) \approx \tilde{u}_N(x),$$

where $F_j(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{x_j\}_{j=0}^N$. Then, the Jacobi collocation methods are to seek $\tilde{u}_N'(x)$, $\tilde{u}_N(x)$ such that $\{u_i'\}_{i=0}^N$, $\{u_i\}_{i=0}^N$ satisfy the following collocation equations:

$$u_i' = \tilde{a}(x_i)u_i + \tilde{b}(x_i) + \left(\frac{1+x_i}{2}\right)^{1-\mu} \sum_{j=0}^N u_j \left(\sum_{k=0}^N \tilde{K}(x_i, \tau(x_i, \theta_k)) F_j(\tau(x_i, \theta_k)) \omega_k \right), \quad (2.15a)$$

$$u_i = u_{-1} + \frac{1+x_i}{2} \sum_{j=0}^N u_j' \left(\sum_{k=0}^N F_j(\tau(x_i, \tilde{\theta}_k)) \tilde{\omega}_k \right). \tag{2.15b}$$

We can get the values of $\{u_i'\}_{i=0}^N$ and $\{u_i\}_{i=0}^N$ by solving the system of linear equations (2.15a)-(2.15b) and obtain the expressions of $\tilde{u}_N'(x)$ and $\tilde{u}_N(x)$ accordingly.

3 Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

Lemma 3.1. (see [9]) *Let $\mathcal{P}_N$ denote the space of all polynomials of degree not exceeding N. Assume that Gauss quadrature formula relative to the Jacobi weight is used to integrate the product $v\phi$, where $v \in H_{\omega^{\alpha,\beta}}^m(-1,1)$ for some $m \geq 1$ and $\phi \in \mathcal{P}_N$. Then there exists a constant C independent of N such that*

$$\left| (v,\phi)_{\omega^{\alpha,\beta}} - (v,\phi)_N \right| \leq CN^{-m} |v|_{H_{\omega^{\alpha,\beta}}^{m;N}(-1,1)} \|\phi\|_{L_{\omega^{\alpha,\beta}}^2(-1,1)}, \tag{3.1}$$

where

$$|v|_{H_{\omega^{\alpha,\beta}}^{m;N}(-1,1)} = \left(\sum_{k=\min(m,N+1)}^m \|v^{(k)}\|_{L_{\omega^{\alpha,\beta}}^2(-1,1)}^2 \right)^{\frac{1}{2}}, \tag{3.2a}$$

$$(v,\phi)_N = \sum_{k=0}^N v(\hat{\theta}_k)\phi(\hat{\theta}_k)\hat{\omega}_k. \tag{3.2b}$$

Lemma 3.2. (see [19]) *Let $\{\hat{F}_j(x)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss points $\{\hat{\theta}_k\}_{k=0}^N$ of the Jacobi polynomials. Then*

$$\|I_N^{\alpha,\beta}\|_\infty := \max_{x\in[-1,1]} \sum_{j=0}^N |\hat{F}_j(x)| = \begin{cases} \mathcal{O}(\log N), & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ \mathcal{O}(N^{\gamma+\frac{1}{2}}), & \gamma = \max(\alpha,\beta), \text{ otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.3. *Assume that $v \in H^m_{\omega^{-\mu,-\mu}}(-1,1)$ and denote $I_N^{-\mu,-\mu}v$ its interpolation polynomial associated with the Jacobi Gauss points $\{x_i\}_{i=0}^N$, namely,*

$$I_N^{-\mu,-\mu}v(x) = \sum_{i=0}^N v(x_i)F_i(x). \tag{3.4}$$

Then the following estimates hold

(1) $\left\| v - I_N^{-\mu,-\mu}v \right\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \leq CN^{-m}|v|_{H^{m;N}_{\omega^{-\mu,-\mu}}(-1,1)}$,

(2) $\left\| v - I_N^{-\mu,-\mu}v \right\|_{L^\infty(-1,1)} \leq \begin{cases} CN^{1-\mu-m}|v|_{H^{m;N}_{\omega^c}(-1,1)}, & 0 < \mu < \frac{1}{2}, \\ CN^{\frac{1}{2}-m}\log N|v|_{H^{m;N}_{\omega^c}(-1,1)}, & \frac{1}{2} \leq \mu < 1, \end{cases}$

where $\omega^c = \omega^{-\frac{1}{2},-\frac{1}{2}}$.

Proof. The inequality (1) can be found in [9]. We only prove (2). Let $I_N^c v \in \mathcal{P}_N$ denote the interpolant of v at Chebyshev Gauss points. From (5.5.28) in [9], the interpolation error estimate in the maximum norm is given by

$$\|v - I_N^c v\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m}|v|_{H^{m;N}_{\omega^c}(-1,1)}. \tag{3.5}$$

Note that

$$I_N^{-\mu,-\mu}p(x) = p(x), \quad \text{i.e.,} \quad (I_N^{-\mu,-\mu} - I)p(x) = 0, \quad \forall p(x) \in \mathcal{P}_N. \tag{3.6}$$

By using (3.6), Lemma 3.2 and (3.5), we obtain that

$$\begin{aligned} &\left\| v - I_N^{-\mu,-\mu}v \right\|_{L^\infty(-1,1)} \\ =&\left\| v - I_N^c v + I_N^{-\mu,-\mu}(I_N^c v) - I_N^{-\mu,-\mu}v \right\|_{L^\infty(-1,1)} \\ \leq&\left\| v - I_N^c v \right\|_{L^\infty(-1,1)} + \left\| I_N^{-\mu,-\mu}(I_N^c v - v) \right\|_{L^\infty(-1,1)} \\ \leq&\left(1 + \left\| I_N^{-\mu,-\mu} \right\|_\infty\right)\left\| v - I_N^c v \right\|_{L^\infty(-1,1)} \\ \leq&\begin{cases} CN^{1-\mu-m}|v|_{H^{m;N}_{\omega^c}(-1,1)}, & 0 < \mu < \frac{1}{2}, \\ CN^{\frac{1}{2}-m}\log N|v|_{H^{m;N}_{\omega^c}(-1,1)}, & \frac{1}{2} \leq \mu < 1. \end{cases} \end{aligned}$$

Hence, the lemma is proved. □

Lemma 3.4. *(Gronwall inequality) If a nonnegative integrable function $E(x)$ satisfies*

$$E(x) \leq L \int_{-1}^x E(s)ds + J(x), \quad -1 < x \leq 1, \tag{3.7}$$

where $J(x)$ is an integrable function, then

$$\|E\|_{L^\infty(-1,1)} \leq C\|J\|_{L^\infty(-1,1)}, \tag{3.8a}$$

$$\|E\|_{L^p_{\omega^{\alpha,\beta}}(-1,1)} \leq C\|J\|_{L^p_{\omega^{\alpha,\beta}}(-1,1)}, \quad p \geq 1. \tag{3.8b}$$

Lemma 3.5. *For nonnegative integer r and $\kappa \in (0,1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\mathcal{T}_N v \in \mathcal{P}_N$ such that*

$$\|v - \mathcal{T}_N v\|_{L^\infty(-1,1)} \leq C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{r,\kappa}, \tag{3.9}$$

where $\|\cdot\|_{r,\kappa}$ is the standard norm in $C^{r,\kappa}([-1,1])$. Actually, as stated in [22, 23], $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}([-1,1])$ into $\mathcal{P}_N$.

Lemma 3.6. (see [12]) *Let $\kappa \in (0,1)$ and $\mathcal{M}$ be defined by*

$$(\mathcal{M}v)(x) = \int_{-1}^{x} (x-\tau)^{-\mu} \tilde{K}(x,\tau) v(\tau) d\tau. \tag{3.10}$$

Then, for any function $v \in C([-1,1])$, there exists a positive constant C such that

$$\frac{|\mathcal{M}v(x') - \mathcal{M}v(x'')|}{|x' - x''|^\kappa} \leq C \max_{x \in [-1,1]} |v(x)|, \tag{3.11}$$

under the assumption that $0 < \kappa < 1 - \mu$, for any $x', x'' \in [-1,1]$ and $x' \neq x''$. This implies that

$$\|\mathcal{M}v\|_{0,\kappa} \leq C \max_{x \in [-1,1]} |v(x)|, \quad 0 < \kappa < 1 - \mu. \tag{3.12}$$

Lemma 3.7. (see [17]) *For all measurable function $f \geq 0$, the following generalized Hardy inequality*

$$\left(\int_a^b |(Tf)(x)|^q u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b |f(x)|^p v(x) dx \right)^{\frac{1}{p}}, \tag{3.13}$$

holds if and only if

$$\sup_{a < x < b} \left(\int_x^b u(t) dt \right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty, \qquad p' = \frac{p}{p-1}, \tag{3.14}$$

for the case $1 < p \leq q < \infty$. Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x,t) f(t) dt,$$

with $k(x,t)$ a given kernel, u, v weight functions and $-\infty \leq a < b \leq \infty$.

Lemma 3.8. (see [21]) *For every bounded function $v(x)$, there exists a constant C independent of v such that*

$$\sup_N \left\| \sum_{j=0}^{N} v(x_j) F_j(x) \right\|_{L^2_{\omega^{\alpha,\beta}}(-1,1)} \leq C \max_{x \in [-1,1]} |v(x)|, \tag{3.15}$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{x_i\}_{i=0}^N$.

4 Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed approximations. Firstly, we will carry our convergence analysis in L^∞ space.

Theorem 4.1. *Let $u(x)$ be the exact solution of the Volterra integro-differential equation (2.6) with (2.7), which is assumed to be sufficiently smooth. Assume that $\tilde{u}'_N(x)$ and $\tilde{u}_N(x)$ are obtained by using the spectral collocation scheme (2.15a)-(2.15b) together with a polynomial interpolation (2.14). If μ associated with the weakly singular kernel satisfies $0 < \mu < 1$ and $u \in H^{m+1}_{\omega^{-\mu,-\mu}}(-1,1)$, then*

$$
\|u - \tilde{u}_N\|_{L^\infty(-1,1)} \leq
\begin{cases}
CN^{\frac{1}{2}-\mu-m}\left(K^* \|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\right), & 0 < \mu < \frac{1}{2}, \\
CN^{-m}\log N\left(K^* \|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\right), & \frac{1}{2} \leq \mu < 1,
\end{cases}
\tag{4.1a}
$$

$$
\|u' - \tilde{u}'_N\|_{L^\infty(-1,1)} \leq
\begin{cases}
CN^{\frac{1}{2}-\mu-m}\left(K^* \|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\right), & 0 < \mu < \frac{1}{2}, \\
CN^{-m}\log N\left(K^* \|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\right), & \frac{1}{2} \leq \mu < 1,
\end{cases}
\tag{4.1b}
$$

provided that N is sufficiently large, where C is a constant independent of N but which will depend on the bounds of the functions $\tilde{a}(x)$, $\tilde{K}(x,\tau)$ and the index μ,

$$
K^* = \max_{x \in [-1,1]} |\tilde{K}(x, \tau(x, \cdot))|_{H^{m;N}_{\omega^{-\mu,0}}(-1,1)},
\tag{4.2a}
$$

$$
U = |u|_{H^{m;N}_{\omega^c}(-1,1)} + |u'|_{H^{m;N}_{\omega^c}(-1,1)}.
\tag{4.2b}
$$

Proof. First, we use the weighted inner product to rewrite (2.10a) as

$$
u'(x_i) = \tilde{a}(x_i)u(x_i) + \tilde{b}(x_i) + \left(\frac{1+x_i}{2}\right)^{1-\mu}\left(\tilde{K}(x_i, \tau(x_i, \cdot)), u(\tau(x_i, \cdot))\right)_{\omega^{-\mu,0}}.
\tag{4.3}
$$

By using the discrete inner product, we set

$$
\left(\tilde{K}(x_i, \tau(x_i, \cdot)), \phi(\tau(x_i, \cdot))\right)_N = \sum_{k=0}^{N} \tilde{K}(x_i, \tau(x_i, \theta_k))\phi(\tau(x_i, \theta_k))\omega_k.
$$

Then, the numerical scheme (2.15a)-(2.15b) can be written as

$$
u'_i = \tilde{a}(x_i)u_i + \tilde{b}(x_i) + \left(\frac{1+x_i}{2}\right)^{1-\mu}\left(\tilde{K}(x_i, \tau(x_i, \cdot)), \tilde{u}_N(\tau(x_i, \cdot))\right)_N,
\tag{4.4a}
$$

$$
u_i = u_{-1} + \int_{-1}^{x_i} \tilde{u}'_N(\tau)d\tau,
\tag{4.4b}
$$

by using the following equality

$$\int_{-1}^{x_i} \tilde{u}_N'(\tau)d\tau = \int_{-1}^{x_i} \sum_{j=0}^{N} u_j' F_j(\tau)d\tau$$

$$= \frac{1+x_i}{2} \int_{-1}^{1} \sum_{j=0}^{N} u_j' F_j(\tau(x_i,\theta))d\theta$$

$$= \frac{1+x_i}{2} \sum_{j=0}^{N} u_j' \left(\sum_{k=0}^{N} F_j(\tau(x_i,\tilde{\theta}_k))\tilde{\omega}_k \right). \tag{4.5}$$

We now subtract (4.4a) from (4.3) and subtract (4.4b) from (2.9b) to get the error equations:

$$u'(x_i) - u_i' = \tilde{a}(x_i)(u(x_i) - u_i) + \left(\frac{1+x_i}{2}\right)^{1-\mu} \left(\tilde{K}(x_i,\tau(x_i,\cdot)), e_u(\tau(x_i,\cdot))\right)_{\omega^{-\mu,0}} + I(x_i)$$

$$= \tilde{a}(x_i)(u(x_i) - u_i) + \int_{-1}^{x_i}(x_i - \tau)^{-\mu}\tilde{K}(x_i,\tau)e_u(\tau)d\tau + I(x_i), \tag{4.6a}$$

$$u(x_i) - u_i = \int_{-1}^{x_i} e_{u'}(\tau)d\tau, \tag{4.6b}$$

where $e_u(x) = u(x) - \tilde{u}_N(x)$, $e_{u'}(x) = u'(x) - \tilde{u}_N'(x)$ are the error functions, and

$$I(x) = \left(\frac{1+x}{2}\right)^{1-\mu} \left(\left(\tilde{K}(x,\tau(x,\cdot)), \tilde{u}_N(\tau(x,\cdot))\right)_{\omega^{-\mu,0}} - \left(\tilde{K}(x,\tau(x,\cdot)), \tilde{u}_N(\tau(x,\cdot))\right)_N \right).$$

Using the integration error estimate in Lemma 3.1, we have

$$|I(x)| \leq CN^{-m}|\tilde{K}(x,\tau(x,\cdot))|_{H^{m;N}_{\omega^{-\mu,0}}(-1,1)} \|\tilde{u}_N(\tau(x,\cdot))\|_{L^2_{\omega^{-\mu,0}}(-1,1)}$$

$$\leq CN^{-m} \max_{x\in[-1,1]} |\tilde{K}(x,\tau(x,\cdot))|_{H^{m;N}_{\omega^{-\mu,0}}(-1,1)} \left(\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + \|e_u\|_{L^\infty(-1,1)} \right). \tag{4.7}$$

Multiplying $F_i(x)$ on both sides of Eqs. (4.6a) and (4.6b) and summing up from $i = 0$ to $i = N$ yield

$$I_N^{-\mu,-\mu}u'(x) - \tilde{u}_N'(x) = I_N^{-\mu,-\mu}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e_u(\tau)d\tau \right)$$

$$+ I_N^{-\mu,-\mu}\left(\tilde{a}(x) \int_{-1}^{x} e_{u'}(\tau)d\tau \right) + J_1(x), \tag{4.8a}$$

$$I_N^{-\mu,-\mu}u(x) - \tilde{u}_N(x) = I_N^{-\mu,-\mu}\left(\int_{-1}^{x} e_{u'}(\tau)d\tau \right), \tag{4.8b}$$

where

$$J_1(x) = \sum_{i=0}^{N} I(x_i)F_i(x).$$

Consequently,

$$e_{u'}(x) = \tilde{a}(x) \int_{-1}^{x} e_{u'}(\tau)d\tau + \int_{-1}^{x} (x-\tau)^{-\mu}\tilde{K}(x,\tau)e_u(\tau)d\tau$$
$$+ J_1(x) + J_2(x) + J_4(x) + J_5(x), \tag{4.9a}$$
$$e_u(x) = \int_{-1}^{x} e_{u'}(\tau)d\tau + J_3(x) + J_6(x), \tag{4.9b}$$

where

$$J_2(x) = u'(x) - I_N^{-\mu,-\mu}u'(x), \qquad J_3(x) = u(x) - I_N^{-\mu,-\mu}u(x),$$
$$J_4(x) = I_N^{-\mu,-\mu}\left(\tilde{a}(x)\int_{-1}^{x} e_{u'}(\tau)d\tau\right) - \tilde{a}(x)\int_{-1}^{x} e_{u'}(\tau)d\tau,$$
$$J_5(x) = I_N^{-\mu,-\mu}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e_u(\tau)d\tau\right) - \int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)e_u(\tau)d\tau,$$
$$J_6(x) = I_N^{-\mu,-\mu}\left(\int_{-1}^{x} e_{u'}(\tau)d\tau\right) - \int_{-1}^{x} e_{u'}(\tau)d\tau.$$

Due to Eqs. (4.9a)-(4.9b) and using the Dirichlet's formula which states

$$\int_{-1}^{x}\int_{-1}^{\tau} \Phi(\tau,s)dsd\tau = \int_{-1}^{x}\int_{s}^{x} \Phi(\tau,s)d\tau ds,$$

provided the integral exists, we obtain

$$e_{u'}(x) = \int_{-1}^{x}\left(\tilde{a}(x) + \int_{\tau}^{x}(x-s)^{-\mu}\tilde{K}(x,s)ds\right)e_{u'}(\tau)d\tau$$
$$+ \int_{-1}^{x}(x-\tau)^{-\mu}\tilde{K}(x,\tau)(J_3(\tau) + J_6(\tau))d\tau$$
$$+ J_1(x) + J_2(x) + J_4(x) + J_5(x). \tag{4.10}$$

Denote $D := \{(x,s) : -1 \le s \le x, x \in [-1,1]\}$, we have

$$\left|\tilde{a}(x) + \int_{\tau}^{x}(x-s)^{-\mu}\tilde{K}(x,s)ds\right| \le \max_{x\in[-1,1]}|\tilde{a}(x)| + \max_{(x,s)\in D}|\tilde{K}(x,s)|\int_{\tau}^{x}(x-s)^{-\mu}ds$$
$$\le \max_{x\in[-1,1]}|\tilde{a}(x)| + \frac{2^{1-\mu}}{1-\mu}\max_{(x,s)\in D}|\tilde{K}(x,s)|$$
$$\triangleq M.$$

Eq. (4.10) gives

$$|e_{u'}(x)| \le M\int_{-1}^{x}|e_{u'}(\tau)|d\tau + \int_{-1}^{x}(x-\tau)^{-\mu}|\tilde{K}(x,\tau)|(|J_3(\tau)| + |J_6(\tau)|)d\tau$$
$$+ |J_1(x)| + |J_2(x)| + |J_4(x)| + |J_5(x)|.$$

It follows from the Gronwall inequality in Lemma 3.4 that

$$\|e_{u'}\|_{L^\infty(-1,1)} \le C \sum_{i=1}^{6} \|J_i\|_{L^\infty(-1,1)}. \tag{4.11}$$

It follows from (4.9b) that

$$\|e_u\|_{L^\infty(-1,1)} \le 2\|e_{u'}\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_6\|_{L^\infty(-1,1)}. \tag{4.12}$$

Using Lemma 3.2, the estimates (4.7) and (4.12), we have

$$\|J_1\|_{L^\infty(-1,1)} \le
\begin{cases}
CN^{\frac{1}{2}-\mu} \max\limits_{0\le i\le N} |I(x_i)|, & 0 < \mu < \frac{1}{2}, \\[2mm]
C\log N \max\limits_{0\le i\le N} |I(x_i)|, & \frac{1}{2} \le \mu < 1,
\end{cases}$$

$$\le
\begin{cases}
CN^{\frac{1}{2}-\mu-m} \max\limits_{x\in[-1,1]} |\tilde{K}(x,\tau(x,\cdot))|_{H^{m;N}_{\omega^{-\mu,0}}(-1,1)} \big(\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} \\[1mm]
\quad +2\|e_{u'}\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_6\|_{L^\infty(-1,1)}\big), & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{-m}\log N \max\limits_{x\in[-1,1]} |\tilde{K}(x,\tau(x,\cdot))|_{H^{m;N}_{\omega^{-\mu,0}}(-1,1)} \big(\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} \\[1mm]
\quad +2\|e_{u'}\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_6\|_{L^\infty(-1,1)}\big), & \frac{1}{2} \le \mu < 1.
\end{cases}$$

Due to Lemma 3.3,

$$\|J_2\|_{L^\infty(-1,1)} \le
\begin{cases}
CN^{1-\mu-m}|u'|_{H^{m;N}_{\omega^c}(-1,1)}, & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{\frac{1}{2}-m}\log N|u'|_{H^{m;N}_{\omega^c}(-1,1)}, & \frac{1}{2} \le \mu < 1,
\end{cases}$$

$$\|J_3\|_{L^\infty(-1,1)} \le
\begin{cases}
CN^{1-\mu-m}|u|_{H^{m;N}_{\omega^c}(-1,1)}, & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{\frac{1}{2}-m}\log N|u|_{H^{m;N}_{\omega^c}(-1,1)}, & \frac{1}{2} \le \mu < 1.
\end{cases}$$

By virtue of Lemma 3.3 (2) with $m = 1$,

$$\|J_4\|_{L^\infty(-1,1)} \le
\begin{cases}
CN^{-\mu}\|e_{u'}\|_{L^\infty(-1,1)}, & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{-\frac{1}{2}}\log N\|e_{u'}\|_{L^\infty(-1,1)}, & \frac{1}{2} \le \mu < 1,
\end{cases}$$

$$\|J_6\|_{L^\infty(-1,1)} \le
\begin{cases}
CN^{-\mu}\|e_{u'}\|_{L^\infty(-1,1)}, & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{-\frac{1}{2}}\log N\|e_{u'}\|_{L^\infty(-1,1)}, & \frac{1}{2} \le \mu < 1.
\end{cases}$$

We now estimate the term $J_5(x)$. It follows from Lemma 3.5 and Lemma 3.6 that

$$\|J_5\|_{L^\infty(-1,1)} = \big\|(I_N^{-\mu,-\mu} - I)\mathcal{M}e_u\big\|_{L^\infty(-1,1)} = \big\|(I_N^{-\mu,-\mu} - I)(\mathcal{M}e_u - \mathcal{T}_N\mathcal{M}e_u)\big\|_{L^\infty(-1,1)}$$

$$\le \big(1 + \|I_N^{-\mu,-\mu}\|_\infty\big)CN^{-\kappa}\|\mathcal{M}e_u\|_{0,\kappa}$$

$$\le
\begin{cases}
CN^{\frac{1}{2}-\mu-\kappa}\|e_u\|_{L^\infty(-1,1)}, & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{-\kappa}\log N\|e_u\|_{L^\infty(-1,1)}, & \frac{1}{2} \le \mu < 1,
\end{cases}$$

$$\le
\begin{cases}
CN^{\frac{1}{2}-\mu-\kappa}\big(2\|e_{u'}\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_6\|_{L^\infty(-1,1)}\big), & 0 < \mu < \frac{1}{2}, \\[2mm]
CN^{-\kappa}\log N\big(2\|e_{u'}\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_6\|_{L^\infty(-1,1)}\big), & \frac{1}{2} \le \mu < 1,
\end{cases}$$

where in the last step we have used Lemma 3.6 under the following assumption:

$$\begin{cases} \frac{1}{2} - \mu < \kappa < 1 - \mu, & \text{when } 0 < \mu < \frac{1}{2}, \\ 0 < \kappa < 1 - \mu, & \text{when } \frac{1}{2} \leq \mu < 1. \end{cases}$$

We now obtain the estimate for $\|e_{u'}\|_{L^\infty(-1,1)}$ by using (4.11):

$$\|e_{u'}\|_{L^\infty(-1,1)} \leq \begin{cases} CN^{\frac{1}{2}-\mu-m}\big(K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\big), & 0 < \mu < \frac{1}{2}, \\ CN^{-m}\log N\big(K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\big), & \frac{1}{2} \leq \mu < 1. \end{cases}$$

The above estimate, together with (4.12), yield

$$\|e_u\|_{L^\infty(-1,1)} \leq \begin{cases} CN^{\frac{1}{2}-\mu-m}\big(K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\big), & 0 < \mu < \frac{1}{2}, \\ CN^{-m}\log N\big(K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + N^{\frac{1}{2}}U\big), & \frac{1}{2} \leq \mu < 1. \end{cases}$$

This completes the proof of the theorem. □

Next, we will give the error estimates in $L^2_{\omega^{-\mu,-\mu}}$ space.

Theorem 4.2. *If the hypotheses given in Theorem 4.1 hold, then*

$$\|u - \tilde{u}_N\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$$
$$\leq \begin{cases} CN^{-m}\big(V_1 + V_2 + N^{1-\mu-\kappa}U + N^{\frac{1}{2}-\mu-\kappa}V_1\big), & 0 < \mu < \frac{1}{2}, \\ CN^{-m}\big(V_1 + V_2 + N^{\frac{1}{2}-\kappa}\log N U + N^{-\kappa}\log N V_1\big), & \frac{1}{2} \leq \mu < 1, \end{cases} \quad (4.13a)$$

and

$$\|u' - \tilde{u}'_N\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$$
$$\leq \begin{cases} CN^{-m}\big(V_1 + V_2 + N^{1-\mu-\kappa}U + N^{\frac{1}{2}-\mu-\kappa}V_1\big), & 0 < \mu < \frac{1}{2}, \\ CN^{-m}\big(V_1 + V_2 + N^{\frac{1}{2}-\kappa}\log N U + N^{-\kappa}\log N V_1\big), & \frac{1}{2} \leq \mu < 1, \end{cases} \quad (4.13b)$$

for any $\kappa \in (0, 1 - \mu)$ provided that N is sufficiently large and C is a constant independent of N, where

$$V_1 = K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)}, \quad (4.14a)$$

$$V_2 = K^*\|u\|_{H^2_{\omega^c}(-1,1)} + |u|_{H^{m;N}_{\omega^{-\mu,-\mu}}(-1,1)} + |u'|_{H^{m;N}_{\omega^{-\mu,-\mu}}(-1,1)}. \quad (4.14b)$$

Proof. By using the Gronwall inequality (Lemma 3.4) and the Hardy inequality (Lemma 3.7), we obtain that

$$\|e_{u'}\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \leq C \sum_{i=1}^{6} \|J_i\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}. \quad (4.15)$$

Now, using Lemma 3.8, we have

$$\|J_1\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \le C \max_{x\in[-1,1]} |I(x)| \le CN^{-m}K^*\Big(\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)} + \|e_u\|_{L^\infty(-1,1)}\Big). \quad (4.16)$$

By the convergence result in Theorem 4.1 ($m = 1$), we have

$$\|e_u\|_{L^\infty(-1,1)} \le C\big(\|u\|_{H^2_{\omega^c}(-1,1)} + \|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)}\big). \quad (4.17)$$

Consequently,

$$\|J_1\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \le CN^{-m}K^*\big(\|u\|_{H^2_{\omega^c}(-1,1)} + \|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)}\big).$$

Due to Lemma 3.3,

$$\|J_2\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \le CN^{-m}|u'|_{H^{m;N}_{\omega^{-\mu,-\mu}}(-1,1)},$$

$$\|J_3\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \le CN^{-m}|u|_{H^{m;N}_{\omega^{-\mu,-\mu}}(-1,1)}.$$

By virtue of Lemma 3.3 (1) with $m = 1$,

$$\|J_4\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \le CN^{-1}\Big|\tilde{a}(x)\int_{-1}^x e_{u'}(\tau)d\tau\Big|_{H^{1;N}_{\omega^{-\mu,-\mu}}(-1,1)} \le CN^{-1}\|e_{u'}\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)},$$

$$\|J_6\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} \le CN^{-1}\|e_{u'}\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}.$$

Finally, it follows from Lemma 3.5 and Lemma 3.8 that

$$\|J_5\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} = \|(I_N^{-\mu,-\mu} - I)\mathcal{M}e_u\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$$
$$= \|(I_N^{-\mu,-\mu} - I)(\mathcal{M}e_u - \mathcal{T}_N\mathcal{M}e_u)\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$$
$$\le \|I_N^{-\mu,-\mu}(\mathcal{M}e_u - \mathcal{T}_N\mathcal{M}e_u)\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)} + \|\mathcal{M}e_u - \mathcal{T}_N\mathcal{M}e_u\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$$
$$\le C\|\mathcal{M}e_u - \mathcal{T}_N\mathcal{M}e_u\|_{L^\infty(-1,1)}$$
$$\le CN^{-\kappa}\|\mathcal{M}e_u\|_{0,\kappa} \le CN^{-\kappa}\|e_u\|_{L^\infty(-1,1)},$$

where, in the last step we used Lemma 3.6 for any $\kappa \in (0, 1 - \mu)$. By the convergence result in Theorem 4.1, we obtain that

$$\|J_5\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$$
$$\le \begin{cases} CN^{1-\mu-m-\kappa}\big(U + N^{-\frac12}K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)}\big), & 0 < \mu < \frac12, \\ CN^{\frac12-m-\kappa}\log N\big(U + N^{-\frac12}K^*\|u\|_{L^2_{\omega^{-\mu,0}}(-1,1)}\big), & \frac12 \le \mu < 1, \end{cases}$$

for N sufficiently large and for any $\kappa \in (0, 1 - \mu)$. The desired estimates (4.13a) and (4.13b) are obtained. □

5 Numerical experiments

Writing $U_N = (u_0, u_1, \cdots, u_N)^T$ and $U'_N = (u'_0, u'_1, \cdots, u'_N)^T$, we obtain the following equations of the matrix form from (2.15a)-(2.15b):

$$U'_N = (A + C)U_N + d_N,$$
$$U_N = U_{-1} + BU'_N,$$

where

$$C = \mathrm{diag}(\tilde{a}(x_0), \tilde{a}(x_1), \cdots, \tilde{a}(x_N)),$$
$$d_N = (\tilde{b}(x_0), \tilde{b}(x_1), \cdots, \tilde{b}(x_N))^T,$$
$$U_{-1} = u_{-1} \times (1, 1, \cdots, 1)^T.$$

The entries of the matrices are given by

$$A_{ij} = \left(\frac{1 + x_i}{2}\right)^{1-\mu} \sum_{k=0}^{N} \tilde{K}(x_i, \tau(x_i, \theta_k)) F_j(\tau(x_i, \theta_k)) \omega_k,$$

$$B_{ij} = \frac{1 + x_i}{2} \sum_{k=0}^{N} F_j(\tau(x_i, \tilde{\theta}_k)) \tilde{\omega}_k.$$

We give the numerical examples to confirm our analysis.

Example 5.1. Consider the weakly singular Volterra integro-differential equation

$$u'(x) = xu(x) + (2 - x)e^{2x} - \frac{4}{3}(x + 1)^{\frac{3}{4}}$$
$$+ \int_{-1}^{x} (x - \tau)^{-\frac{1}{4}} e^{-2\tau} u(\tau) d\tau, \quad x \in [-1, 1], \tag{5.1a}$$
$$u(-1) = e^{-2}. \tag{5.1b}$$

The corresponding exact solution is given by $u(x) = e^{2x}$. Table 1 shows the errors of approximate solution in L^∞ and weighted L^2 norms obtained by using the spectral methods described above. Furthermore, we also compute the errors of approximate derivative, the results are shown in Table 2. It is observed that the desired exponential

Table 1: Example 5.1: The errors $\|u - \tilde{u}_N\|_{L^\infty(-1,1)}$ and $\|u - \tilde{u}_N\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$.

N	2	4	6	8	10
L^∞-error	1.1653e+001	7.6906e-002	1.3489e-003	2.0320e-005	1.9853e-007
$L^2_{\omega^{-\mu,-\mu}}$-error	9.5006e+000	5.6340e-002	2.9585e-004	1.4944e-006	9.2708e-009
N	12	14	16	18	20
L^∞-error	1.3567e-009	6.8411e-012	6.1994e-013	6.9277e-013	5.5955e-013
$L^2_{\omega^{-\mu,-\mu}}$-error	4.8484e-011	4.1311e-013	3.9540e-013	4.5287e-013	3.5640e-013

Table 2: Example 5.1: The errors $\|u' - \tilde{u}'_N\|_{L^\infty(-1,1)}$ and $\|u' - \tilde{u}'_N\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$.

N	2	4	6	8	10
L^∞-error	1.5940e+001	1.3062e-001	3.3346e-003	4.9608e-005	4.7920e-007
$L^2_{\omega^{-\mu,-\mu}}$-error	1.3666e+001	8.4228e-002	4.8786e-004	3.3432e-006	2.3829e-008
N	12	14	16	18	20
L^∞-error	3.2418e-009	1.6200e-011	8.6330e-013	9.3436e-013	7.5672e-013
$L^2_{\omega^{-\mu,-\mu}}$-error	1.3113e-010	7.3088e-013	5.2964e-013	5.9267e-013	4.6692e-013

rate of convergence is obtained. Fig. 1 presents the approximate and exact solution on left side and presents the approximate and exact derivative on right side, which are found in excellent agreement. In Fig. 2, the numerical errors $u - \tilde{u}_N$ and $u' - \tilde{u}'_N$ are plotted for $2 \leq N \leq 20$ in both L^∞ and $L^2_{\omega^{-\mu,-\mu}}$ norms.

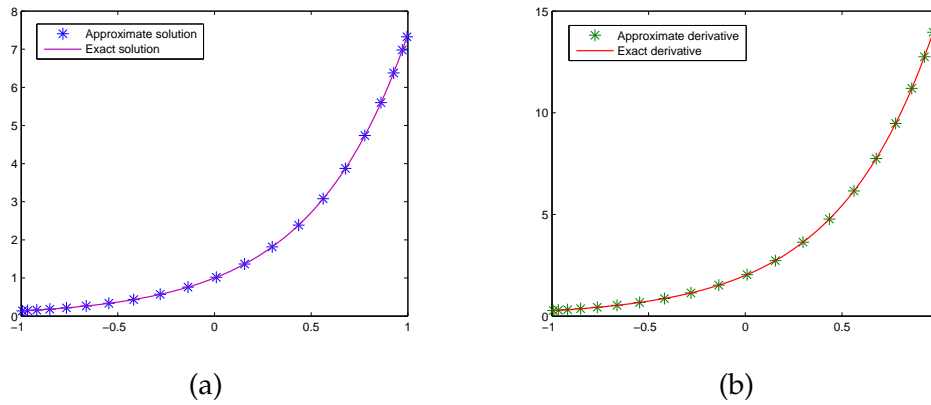(a)   (b)

Figure 1: Example 5.1: Comparison between approximate solution $\tilde{u}_N$ and exact solution u (a); Comparison between approximate derivative $\tilde{u}'_N$ and exact derivative u' (b).

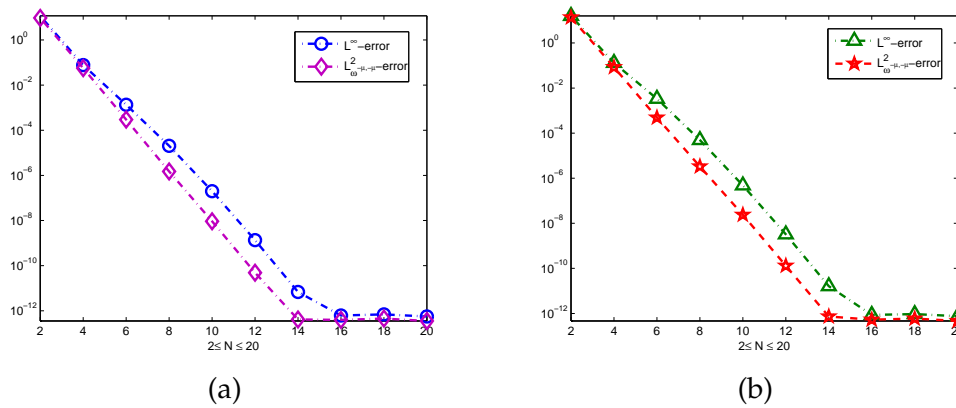(a)   (b)

Figure 2: Example 5.1: The errors $u - \tilde{u}_N$ (a) and $u' - \tilde{u}'_N$ (b) versus the number of collocation points in L^∞ and $L^2_{\omega^{-\mu,-\mu}}$ norms.

Example 5.2. Consider the weakly singular Volterra integro-differential equation

$$w'(x) = 2xw(x) + \cos x - 2x \sin x - \frac{9}{4}(x+1)^{\frac{4}{3}} + 3(x+1)^{\frac{1}{3}}$$

$$+ \int_{-1}^{x} (x-\tau)^{-\frac{2}{3}} \frac{\tau}{\sin \tau} w(\tau) d\tau, \quad x \in [-1,1], \tag{5.2a}$$

$$w(-1) = \sin(-1). \tag{5.2b}$$

The corresponding exact solution is given by $w(x) = \sin x$. The approximate solution $\tilde{w}_N(x)$ and exact solution $w(x)$ (also approximate derivative $\tilde{w}'_N(x)$ and exact derivative $w'(x)$) for (5.2a)-(5.2b) are displayed in Fig. 3. Fig. 4 plots the errors $w - \tilde{w}_N$ and $w' - \tilde{w}'_N$ for $2 \leq N \leq 16$ in both L^∞ and $L^2_{\omega^{-\mu,-\mu}}$ norms. Moreover, the corresponding errors with several values of N are displayed in Table 3 for $w - \tilde{w}_N$ and in Table 4 for $w' - \tilde{w}'_N$.
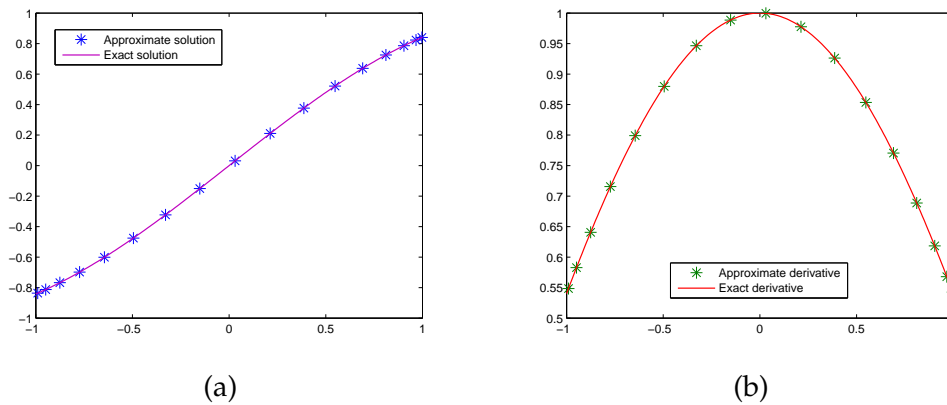

Figure 3: Example 5.2: Comparison between approximate solution $\tilde{w}_N$ and exact solution w (a); Comparison between approximate derivative $\tilde{w}'_N$ and exact derivative w' (b).
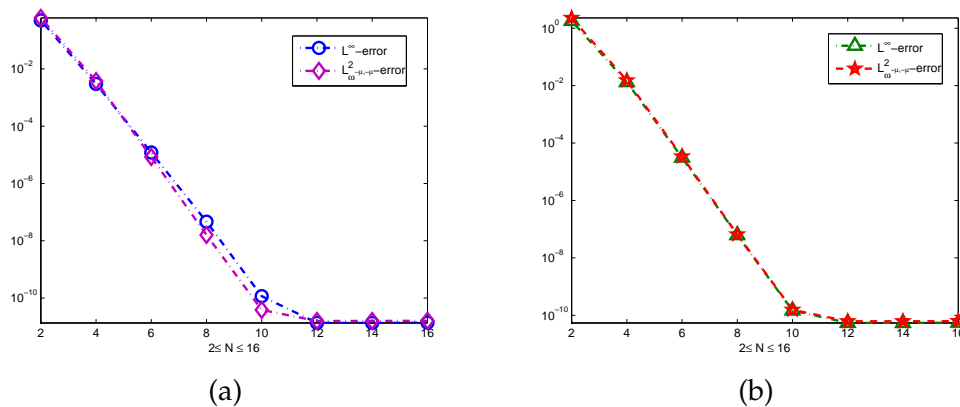

Figure 4: Example 5.2: The errors $w - \tilde{w}_N$ (a) and $w' - \tilde{w}'_N$ (b) versus the number of collocation points in L^∞ and $L^2_{\omega^{-\mu,-\mu}}$ norms.

Table 3: Example 5.2: The errors $\|w - \tilde{w}_N\|_{L^\infty(-1,1)}$ and $\|w - \tilde{w}_N\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$.

N	2	4	6	8
L^∞-error	4.9412e-001	3.0024e-003	1.2020e-005	4.6504e-008
$L^2_{\omega^{-\mu,-\mu}}$-error	6.0446e-001	3.7871e-003	8.4039e-006	1.6011e-008
N	10	12	14	16
L^∞-error	1.1610e-010	1.3630e-011	1.3553e-011	1.3802e-011
$L^2_{\omega^{-\mu,-\mu}}$-error	3.8618e-011	1.5909e-011	1.5835e-011	1.6126e-011

Table 4: Example 5.2: The errors $\|w' - \tilde{w}'_N\|_{L^\infty(-1,1)}$ and $\|w' - \tilde{w}'_N\|_{L^2_{\omega^{-\mu,-\mu}}(-1,1)}$.

N	2	4	6	8
L^∞-error	1.7854e+000	1.3253e-002	3.0984e-005	6.1560e-008
$L^2_{\omega^{-\mu,-\mu}}$-error	2.2852e+000	1.5161e-002	3.3965e-005	6.5357e-008
N	10	12	14	16
L^∞-error	1.4589e-010	5.4296e-011	5.4073e-011	5.5045e-011
$L^2_{\omega^{-\mu,-\mu}}$-error	1.5531e-010	6.1852e-011	6.1581e-011	6.2685e-011

6   Conclusions and future work

This paper proposes a spectral method for first order Volterra integro-differential equations which contain a weakly singular kernel $(t - s)^{-\mu}$ with $0 < \mu < 1$. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of spectral approximations decay exponentially in both infinity and weighted norms, which is a desired feature for a spectral method.

We only investigated the case when the solution is smooth in the present work, with the availability of this methodology, it will be possible to extend the results of this paper to the weakly singular VIDEs with nonsmooth solutions which will be the subject of our future work.

Acknowledgments

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