Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **10**, No. 4, pp. 797-818

DOI: 10.4208/aamm.OA-2017-0223
August 2018

An EDG Method for Distributed Optimal Control of Elliptic PDEs

Xiao Zhang[1], Yangwen Zhang[2] and John Singler[2,*]

[1] *College of Mathematics, Sichuan University, Chengdu 610065, China*
[2] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409, USA*

Received 5 September 2017; Accepted (in revised version) 23 February 2018

Abstract. We consider a distributed optimal control problem governed by an elliptic PDE, and propose an embedded discontinuous Galerkin (EDG) method to approximate the solution. We derive optimal a priori error estimates for the state, dual state, and the optimal control, and suboptimal estimates for the fluxes. We present numerical experiments to confirm our theoretical results.

AMS subject classifications: 65N30, 49M25

Key words: Distributed optimal control, elliptic PDEs, embedded discontinuous Galerkin method, error analysis.

1 Introduction

We consider approximating the solution of the following distributed control problem. Let $\Omega\subset\mathbb{R}^d$ $(d\geq 2)$ be a Lipschitz polyhedral domain with Lipschitz boundary $\Gamma=\partial\Omega$. The goal is to minimize

$$J(u)=\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\|u\|^2_{L^2(\Omega)},\quad \gamma>0, \tag{1.1}$$

subject to

$$-\Delta y=f+u \qquad \text{in } \Omega, \tag{1.2a}$$
$$y=g \qquad \text{on } \partial\Omega. \tag{1.2b}$$

*Corresponding author.
Emails: zhangxiaofem@163.com (X. Zhang), ywzfg4@mst.edu (Y. W. Zhang), singlerj@mst.edu (J. Singler)

It is well known that the optimal control problem (1.1)-(1.2) is equivalent to the optimality system

$$
\begin{aligned}
&-\Delta y = f + u && \text{in } \Omega, && \text{(1.3a)}\\
&y = g && \text{on } \partial\Omega, && \text{(1.3b)}\\
&-\Delta z = y_d - y && \text{in } \Omega, && \text{(1.3c)}\\
&z = 0 && \text{on } \partial\Omega, && \text{(1.3d)}\\
&z - \gamma u = 0 && \text{in } \Omega. && \text{(1.3e)}
\end{aligned}
$$

Different numerical methods for optimal control problems governed by partial differential equations have been extensively studied by many researchers. Numerical methods that have been investigated for this kind of problem include approaches based on standard finite element methods [1,7,15,19,28], mixed finite elements [4–6,8,9,21,22], and discontinuous Galerkin (DG) methods [27,36].

Recently, hybridizable discontinuous Galerkin (HDG) methods have been developed for many partial differential equations; see, e.g., [2,3,10,11,13,14,29–31,34]. HDG methods keep the advantages of DG methods and mixed methods, while also having less globally coupled unknowns. HDG methods have now also been applied to many different optimal control problems [23–25,37].

The embedded discontinuous Galerkin (EDG) methods, originally proposed in [20], are obtained from HDG methods by replacing the discontinuous finite element space for the numerical traces with a continuous space. EDG methods also retain many of the advantages of DG and mixed methods; furthermore, the number of degrees of freedoms for the EDG method are much smaller than the HDG methods. This gain in computational efficiency can come with a loss: for the Poisson equation, convergence rates for the EDG method are one order lower than the HDG method and the convergence rate for the flux is suboptimal [12]. However, for problems with strong convection the enhanced convergence properties of HDG methods are reduced [18]. Therefore, EDG methods are competitive for such problems, and researchers have recently begun to thoroughly investigate EDG methods for various partial differential equations [16,17,26,32,33].

Our long term goal is to devise efficient and accurate methods for complicated optimal flow control problems. EDG methods have potential for such problems; therefore, as a first step, we consider an EDG method to approximate the solution of the above optimal control problem for the Poisson equation. We use an EDG method with polynomials of degree $k \geq 1$ to approximate all the variables of the optimality system (1.3), i.e., the state y, dual state z, the numerical traces, and the fluxes $\boldsymbol{q} = -\nabla y$ and $\boldsymbol{p} = -\nabla z$. We describe the method in Section 2, and in Section 3 we obtain the error estimates

$$
\begin{aligned}
&\|y - y_h\|_{0,\Omega} = \mathcal{O}(h^{k+1}), && \|z - z_h\|_{0,\Omega} = \mathcal{O}(h^{k+1}),\\
&\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,\Omega} = \mathcal{O}(h^{k}), && \|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} = \mathcal{O}(h^{k}),
\end{aligned}
$$