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Efficient Stochastic Runge-Kutta Methods for Stochastic Differential Equations with Small Noises

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Abstract. New stochastic Runge-Kutta (SRK) methods for solving the Itô and Stratonovich stochastic differential equations (SDEs) with small noises are introduced. These SRK methods contain some multiple stochastic integrals simulated easily and have high global mean-square error accuracy. To simplify the calculation process, the stochastic rooted tree analysis is developed to estimate the local error and the global mean-square error estimate for a general class of SRK methods is given. Various improved SRK methods for the Itô or Stratonovich SDEs with non-commutative, commutative, diagonal, scalar, additive or colored small noises are proposed in turn. Finally, the proposed new SRK methods are examined by four test equations and all of the numerical results show the high efficiency of our methods.

AMS subject classifications: 60H35, 60H10, 65L06, 65L20

Key words: Stochastic differential equations, stochastic Runge-Kutta methods, small noises.

1 Introduction

As everyone knows, stochastic differential equations (SDEs) have been widely applied to model many stochastic problems, see, e.g., [20, 25]. Unfortunately, the explicit solutions of most SDEs can not be found. This means that numerical methods are necessary for solving these SDEs. In recent years, various numerical methods have been proposed for the SDEs. For example, some methods of strong convergence can be found in [9, 13, 15, 19, 20, 26, 39, 41, 42, 44, 45, 47], some methods of weak convergence can be found in [1, 2, 17, 20, 21, 26, 36, 40]. However, up to now, the methods with strong order 0.5 are still the most widely used methods for the general SDEs with non-commutative noise, see, e.g., [16,18,23,24,46]. One of the main reasons is that there is no efficient approximate

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methods for some multiple Itô or Stratonovich stochastic integrals used in the numerical methods with high strong order. For example, if a numerical method with strong order 1.0 for the SDEs with non-commutative noise need to be constructed, we must use some multiple stochastic integrals, which will lead to a lot of computational effort.

In many practical models, the intensity of noises is small, but the noises can affect the behaviors of the solutions, especially for the nonlinear systems. Electrical circuit oscillation problem is one of the most common examples that can be described by some SDE systems with small noises, see, e.g., [12, 35]. A similar situation can be found in the problem of signal propagation in neurons [14]. And readers can refer to [38] for more applications of the small noises. In recent years, some efficient numerical methods without using the multiple stochastic integrals simulated difficultly have been obtained to solve the SDEs with small noises.

Milstein and Tretyakov [27, 28] systematically considered the SDEs with small noises and a small parameter $\varepsilon>0$ has been introduced into the diffusion coefficient to facilitate the description of these SDEs. In [27], based on the stochastic Taylor expansion of exact solution, a lot of stochastic Taylor methods and some stochastic Runge-Kutta-type methods with calculation of certain derivatives were proposed and their global mean-square errors can be written as the form $\mathcal{O}(\sum_{i=0}^{r}\varepsilon^i h^{p_i})$, where $p_0, r\in\mathbb{N}$, $p_i\in\frac{1}{2}\mathbb{N}$, $p_{i-1}>p_i$, $i=1,2,\cdots,r$ ($\mathbb{N}$ is the set of non-negative integers). Obviously, the term $\mathcal{O}(h^{p_0})$ will be the dominating error term if the stepsize h is relatively large and the term $\mathcal{O}(\varepsilon^i h^{p_i})$, $i=1,2,\cdots,r$ can take obvious effect on the global mean-square error only when the stepsize h is sufficiently small. This means that the convergence order of the methods in [27] is same as the order of the deterministic part of them when ε is sufficiently small and h is relatively large.

The Itô SDEs with small noises were also considered in [3–6]. In [4, 5], the stochastic linear multistep methods with Maruyama term were proposed and their global mean-square error is $\mathcal{O}(h^{p_0}+\varepsilon h+\varepsilon^2 h^{\frac{1}{2}})$, $p_0\geq 2$. In [6], the improved stochastic linear multistep methods containing some mixed classical-stochastic integrals and certain derivatives were considered and the global mean-square error can be improved to $\mathcal{O}(h^{p_0}+\varepsilon h^2+\varepsilon^2 h^{\frac{1}{2}})$, $p_0\geq 3$. In [3], the stochastic Runge-Kutta (SRK) methods were used to solve the Itô SDEs with small noises, where no derivatives need to be calculated. In particular, the SRK methods with Maruyama term were proposed and their global mean-square error is $\mathcal{O}(h^{p_0}+\varepsilon h+\varepsilon^2 h^{\frac{1}{2}})$, $p_0\geq 2$. Moreover, the improved SRK methods containing some mixed classical-stochastic integrals were also obtained and their global mean-square error is $\mathcal{O}(h^{p_0}+\varepsilon h^2+\varepsilon^2 h^{\frac{1}{2}})$, $p_0\geq 3$. In addition, some adaptive stepsize algorithms of Euler Maruyama methods and SRK methods with Maruyama term were introduced in [29] and [37], respectively.

In the present paper, to improve further the accuracy of the global mean-square error of the SRK methods for the Itô and Stratonovich SDEs with small noises, we add some multiple stochastic integrals simulated easily into the new SRK methods. Based on a different approach as that in [3], we develop the stochastic rooted tree analysis introduced