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Convergence Accelerating in the Homotopy Analysis Method: a New Approach

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Abstract. One of the flash analytic approximate methods of nowadays for the solution of highly nonlinear algebraic or differential equations arising from the mathematical modelling of industrial and technological applications is the homotopy analysis method (HAM). The success of the HAM is mainly due to a so-called convergence control parameter, h, plugged into the system externally which is missing in other competing methods. A simple algorithm to determine this parameter is introduced in this paper, besides the well-known approaches of constant $h-$level curves, the squared residual error and the recent ratio technique. Comparison of the four approaches yields nearly the same convergence control parameters with the advantage of the newly proposed approach in terms of its simplicity and less CPU time requirement. Moreover, a convergence accelerating method is suggested here based on updating the initial guess of the solution at some low-order homotopy series approximation of the solution. It appears to extend the interval of convergence control parameter. The provided examples of real life phenomena in combination with this technique demonstrate a successful improvement over the classical HAM method.

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Key words: Homotopy analysis method, nonlinear system, convergence control parameter, accelerating the convergence.

1 Introduction

The last two decades researches have proven that the homotopy analysis method (HAM) is a powerful candidate for the solution of strongly nonlinear algebraic and differential type equations resulting from mathematical modelling of the real-life physical phenomena [15–17]. The success of the HAM highly depends upon the so-called convergence control parameter h embedded externally into the system. Actually, existence of such a parameter mainly distinguishes the HAM from its variants, as summarized in the recent

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book [17]. The preliminaries of the current work are to introduce an effective way of determining this parameter from a simple and quick approach and also to introduce a convergence accelerating technique of the traditional HAM.

The homotopy analysis method is based on the methodology of homotopy in topology. The idea is to approximate a nonlinear solution by composing infinitely many linear sub solutions. Making use of such a conceptual approach, a great deal of nonlinear equations was treated, all of which can not be cited in this limited space. However, some recent important applications of the HAM to science and engineering problems may be found in the publications [1, 8, 14, 23–26]. For the convergence issue of the HAM method, the article [27] may be referred. Particularly, the "steady-state" resonant waves not detected by the traditional analytic methods in the past 60 years were successfully discovered by HAM in [18], whose physical existence was also approved via a recent experiment in [19]. Some recent applications of HAM include the effectively solving the algebraic equations in low iterations [5], analytically approximating the dual solutions in mixed convection flow occurring in a semi-infinite domain [3], obtaining analytical estimates for the system of nonlinear differential equations [30], solving partial differential equations [9, 10] and treating fuzzy fractional [11], other fractional [7] and singular Lane Emden type equations [4].

In [28], the authors gave a convergence criterion for the variant of HAM, namely the homotopy perturbation method. The weakness of the method in terms of convergency was exemplified in [12]. However, by an optimal homotopy analysis approach proposed in [13], the convergence of the series solution via HAM method to the strongly nonlinear differential equations can always be ensured. The idea of [13] is to determine the optimum h by minimizing the squared residual error. Minimization of the norm of a discrete residual function, systematically, at each order of HAM approximation was also implemented in [2]. This idea was also recently adopted in the Adomian Decomposition Method [30]. Another most commonly used notion, $h-$level curve analysis, is to plot an unknown physical quantity versus the convergence control parameter h, and explore the interval over which the least variation in the physical quantity is observed. A third approach was recently proposed by Turkyilmazoglu, see Chapter 5 in the book [17] and [29]. Since the HAM method produces a series solution, the successive ratio of the terms under the guidance of a proper norm was undertaken in the latter and by requiring the ratio to be as small as possible resulting in the fastest convergence rate and hence optimum convergence control parameter.

The motivation behind the present research is to supply another way of determination of convergence control parameter and also to seek further convergence accelerating techniques. In parallel to the first objective, a simple and effective approach is proposed leading to almost the same optimal convergence control parameter as for the above mentioned two methods, yet demanding less computational effort and time. The second objective is achieved by updating the initial guess to the solution at some low approximation of homotopy series and then carrying out the HAM method. An extended interval of convergence control parameter results under the accelerating scheme. Examples from