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Some Weighted Averaging Methods for Gradient Recovery

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Abstract. We propose some new weighted averaging methods for gradient recovery, and present analytical and numerical investigation on the performance of these weighted averaging methods. It is shown analytically that the harmonic averaging yields a superconvergent gradient for any mesh in one-dimension and the rectangular mesh in two-dimension. Numerical results indicate that these new weighted averaging methods are better recovered gradient approaches than the simple averaging and geometry averaging methods under triangular mesh.

AMS subject classifications: 65N12, 65N15, 65N30, 65D10, 74S05, 41A10, 41A25

Key words: Finite element method, weighted averaging, gradient recovery.

1 Introduction

Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to obtain improved solutions. A classical recovery technique is a simple averaging technique which is as old as the finite element method itself. Whereafter, different kinds of post-processing techniques are developed based on weighted averaging [4–7, 11], local or global projections [3, 8, 13, 17], post-processing interpolation [14, 22], smoothing techniques [9, 10, 18, 20], and the local least-squares methods including the superconvergent patch recovery (SPR) [25–27], the polynomial preserving recovery (PPR) [16, 23] and the superconvergent cluster recovery (SCR) [12].

For the Lagrange element, the gradient of the finite element approximation provides a discontinuous approximation. Two classical techniques of simple averaging

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and geometry averaging are deveopled by engineers to improve the precision of finite element solution. Those well-known recovery techniques construct an gradient approximation for each node by averaging the contributions of the surrounding nodes. These values may be interpolated to obtain a continuous approximation over the whole domain.

In this paper, we propose some new weighted averaging techniques which are named harmonic averaging, angle averaging, and distant averaging for gradient recovery. Together with simple averaging and geometry averaging, we investigate the relationship of all the five weighted averaging methods. We will show analytically that, for any mesh in one-dimension and rectangular mesh in two-dimension, harmonic averaging is a superconvergent gradient recovery method. Under the triangular mesh, we also provide numerical evidence to show that harmonic averaging, angle averaging and distant averaging are performance better than simple averaging and geometry averaging.

The rest of the paper is organized as follows: in Section 2 we describe the construction of these weighted averaging techniques in detail. We analyze harmonic averaging, simple averaging and geometry averaging for one-dimension problems in Section 3, and in Section 4, we investigate harmonic averaging, simple averaging and geometry averaging for two-dimension problems under rectangular mesh. In Section 5, we consider the weighted averaging methods under triangular mesh. Numerical tests illustrating the performance of our new weighted averaging methods are presented in Section 6. Finally, in Section 7, some conclusions and future work are presented.

2 Weighted averaging gradient recovery operators

In this section, we give the definitions of all the five weighted averaging gradient recovery operators including two known called simple averaging and geometry averaging and three new called harmonic averaging, angle averaging and distant averaging. In detail, the simple averaging, geometry averaging and harmonic averaging are defined for both one-dimension problems and two-dimension problems, and angle averaging and distant averaging are only defined for triangular element.

Let $\mathcal{T}_h$ be partition of $\Omega \subset \mathbb{R}^d$ with $d = 1,2$ and S_h be a $\mathcal{C}^0$ finite element space over $\mathcal{T}_h$, and $S_h^d = \prod_{i=1}^d S_h$. Given a finite element function $v \in S_h$, v is piecewise continuous. We firstly define $R_h v$ at each node, where operator $R_h : S_h \to S_h^d$. After defining values of $R_h v$ at all nodes, we obtain $R_h v \in S_h^d$ on the whole domain by interpolation using the original nodal shape functions of S_h.

Firstly, we give the definition of simple averaging, geometry averaging and harmonic averaging in 1D. We consider the problem on the unit interval $I = (0,1)$. Any other interval can be mapped to unit interval by a linear transformation. A subdivision of domain I

$$0 = x_0 < x_1 < x_2 < \cdots < x_N = 1,$$

divides I into N elements $I_i=(x_{i-1},x_i)$ with element size

$$h_i=x_i-x_{i-1},\quad i=1,2,\cdots,N,\qquad h=\max_{1\le i\le N}(x_i-x_{i-1}).$$

Let $\mathcal{N}_h$ denotes the set of the mesh nodes, and S_h be the linear finite element space on the above partition. For any $v\in S_h$, we further denote

$$x_{i-\frac{1}{2}}=\frac{1}{2}(x_{i-1}+x_i),\qquad v_i=v(x_i),\qquad v'_{i-\frac{1}{2}}=v'(x_{i-\frac{1}{2}}).$$

We remark that, recovery points belonging to the domain boundary need special attention, so here for the convenience of analysis and comparison, we limit our consideration on the interior points.

For an inner vertex $x_i\in\mathcal{N}_h$, let $\mathcal{K}_i=I_i\cup I_{i+1}$ denote its corresponding patch. Simple averaging, geometry averaging, and harmonic averaging are defined as follows:

Simple averaging:

$$(S_hv)(x_i):=\frac{1}{2}\big(v'_{i-\frac{1}{2}}+v'_{i+\frac{1}{2}}\big)=\frac{1}{2}\Big(\frac{v_i-v_{i-1}}{h_i}+\frac{v_{i+1}-v_i}{h_{i+1}}\Big).\tag{2.1}$$

Geometry averaging:

$$(G_hv)(x_i):=\frac{h_i}{h_i+h_{i+1}}v'_{i-\frac{1}{2}}+\frac{h_{i+1}}{h_i+h_{i+1}}v'_{i+\frac{1}{2}}=\frac{v_{i+1}-v_{i-1}}{h_i+h_{i+1}}.\tag{2.2}$$

Harmonic averaging:

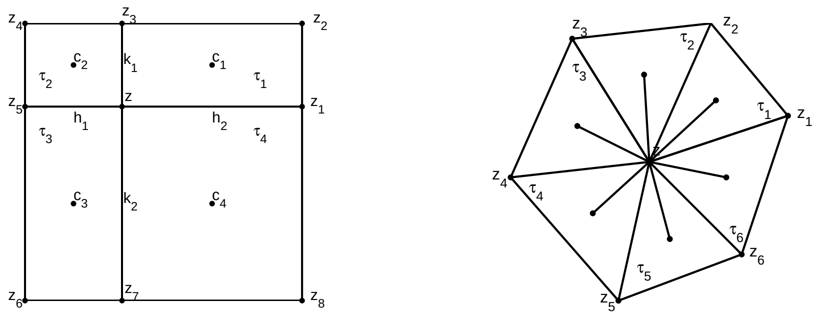
$$\begin{aligned}(H_hv)(x_i):=&\frac{1/h_i}{1/h_i+1/h_{i+1}}v'_{i-\frac{1}{2}}+\frac{1/h_{i+1}}{1/h_i+1/h_{i+1}}v'_{i+\frac{1}{2}}\\=&\frac{h_{i+1}}{h_i+h_{i+1}}v'_{i-\frac{1}{2}}+\frac{h_i}{h_i+h_{i+1}}v'_{i+\frac{1}{2}}\\=&\frac{h_{i+1}}{h_i+h_{i+1}}\cdot\frac{v_i-v_{i-1}}{h_i}+\frac{h_i}{h_i+h_{i+1}}\cdot\frac{v_{i+1}-v_i}{h_{i+1}}.\end{aligned}\tag{2.3}$$

The weights of harmonic averaging were used by Chen and Huang [7]. Here, we derive it through the concept of harmonic, and also extend it to the high dimension problems, we will compare these weighted averaging methods for finite element gradient recovery and present analytical investigation in the performance of them.

Secondly, we introduce the above mentioned three weighted averaging techniques for the rectangular mesh in two-dimension. Let $\mathcal{T}_h$ be a rectangular partition of $\Omega\subset\mathbb{R}^2$ and S_h be a bilinear finite element space over $\mathcal{T}_h$ defined by

$$S_h=\big\{v\in H^1(\Omega):v\in Q_1(\tau),\ \forall\tau\in\mathcal{T}_h\big\},$$

where $Q_k(D)$ denotes the set of polynomials defined in $D\in\mathbb{R}^2$ with degree no more than k in each variable. For a inner node z, let $\mathcal{K}_z$ denote its element patch which



(a) Rectangular element (b) Triangular element

Figure 1: Parameters associated with the node z.

contains all elements that share z as a vertex. In Fig. 1, we denote by $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathcal{K}_z \cap \mathcal{T}_h$ the four rectangles around the considered point z, and $z_1, z_2, \cdots, z_8 \in \mathcal{K}_z$, oriented counterclockwise, are the mesh nodes. Let c_i be the center of rectangle τ_i, and S_i is the area of τ_i, $h = \max\{h_1, h_2, k_1, k_2\}$. As ∇u_h is variable on each rectangular elements, two cases on selecting the sampling points for weighted averaging methods are consider. One is the considered point z itself, and the other is the centers c_i. For any $v \in S_h$, the three weighted averaging methods are defined as follows:

Simple averaging:

$$(S_h v)(z) := \sum_{i=1}^{4} \frac{1}{4} \nabla v(z)|_{\tau_i} \quad \text{or} \quad (S_h v)(z) := \sum_{i=1}^{4} \frac{1}{4} \nabla v(c_i). \tag{2.4}$$

Geometry averaging:

$$(G_h v)(z) := \sum_{i=1}^{4} \frac{S_i}{\sum_{j=1}^{4} S_j} \nabla v(z)|_{\tau_i} \quad \text{or} \quad (G_h v)(z) := \sum_{i=1}^{4} \frac{S_i}{\sum_{j=1}^{4} S_j} \nabla v(c_i). \tag{2.5}$$

Harmonic averaging:

$$(H_h v)(z) := \sum_{i=1}^{4} \frac{1/S_i}{\sum_{j=1}^{4} 1/S_j} \nabla v(z)|_{\tau_i} \quad \text{or} \quad (H_h v)(z) := \sum_{i=1}^{4} \frac{1/S_i}{\sum_{j=1}^{4} 1/S_j} \nabla v(c_i). \tag{2.6}$$

Finally, we give the definitions of all the five weighted averaging operators for the triangular element. Let $\mathcal{T}_h$ be a triangular partition of $\Omega \subset \mathbb{R}^2$ and S_h be a linear finite element space over $\mathcal{T}_h$ defined by

$$S_h = \{v \in H^1(\Omega) : v \in P_1(\tau),\ \ \forall \tau \in \mathcal{T}_h\},$$

where $P_k(D)$ denotes the set of polynomials defined in $D \in \mathbb{R}^2$ with degree no more than k. For a vertex z, let $\mathcal{K}_z$ denote its corresponding patch. In the patch $\mathcal{K}_z$, let $\tau_1, \tau_2, \cdots, \tau_m \in \mathcal{T}_h$ denote the triangles, $c_i, S_i, i = 1, 2, \cdots, m$, denote the centroid and area of τ_i, respectively, $d_i = \|z - c_i\|$, $i = 1, 2, \cdots, m$, denote the distant between the point z and c_i, and $\alpha_i, i = 1, 2, \cdots, m$, denote the angle of τ_i with z as its vertex. For any $v \in S_h$, the five weighted averaging recovery operators are defined as follows:

Simple averaging:

$$S_h v(z) := \sum_{i=1}^{m} \frac{1}{m} \nabla v\big|_{\tau_i}. \tag{2.7}$$

Geometry averaging:

$$G_h v(z) := \sum_{i=1}^{m} \frac{S_i}{\sum_{j=1}^{m} S_j} \nabla v\big|_{\tau_i}. \tag{2.8}$$

Harmonic averaging:

$$H_h v(z) := \sum_{i=1}^{m} \frac{1/S_i}{\sum_{j=1}^{m} 1/S_j} \nabla v\big|_{\tau_i}. \tag{2.9}$$

Angle averaging:

$$A_h v(z) := \sum_{i=1}^{m} \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \nabla v\big|_{\tau_i}. \tag{2.10}$$

Distant averaging:

$$D_h v(z) := \sum_{i=1}^{m} \frac{1/d_i}{\sum_{j=1}^{m} 1/d_j} \nabla v\big|_{\tau_i}. \tag{2.11}$$

The weights of distant averaging are the same as given in [15], they also show that the exact reconstruction of the gradient associated with a quadratic solution on non-uniform meshes in one dimension.

From the definition of simple averaging, geometry averaging, harmonic averaging, angle averaging and distant averaging, we have the following observation:

- Under the uniform mesh, simple averaging, geometry averaging and harmonic averaging are the same, especially for the criss-cross pattern and union-jack pattern, all the five weighted averaging are the same.
- For the rectangular element, we also can define angle averaging and distant averaging, but in this case, angle averaging is the same as simple averaging.

3 Weighted averaging gradient recovery in 1D

In this section, we consider simple averaging, geometry averaging and harmonic averaging which have been defined in Section 2 for gradient recovery with the following two-point boundary value problem

$$\begin{cases} -\dfrac{d}{dx}\Big(a(x)\dfrac{du}{dx}\Big) + b(x)u = f, \quad x \in (0,1), \\ u(0) = 0, \qquad u(1) = 0. \end{cases} \tag{3.1}$$

We assume that a, b and f are sufficiently smooth and that

$$0 < a_0 \leq a(x) \leq a_1 < \infty, \quad 0 \leq b(x) \leq b_1 < \infty, \quad x \in [0,1]. \tag{3.2}$$

A subdivision of domain I

$$0 = x_0 < x_1 < x_2 < \cdots < x_N = 1,$$

divides I into N elements $I_i = (x_{i-1}, x_i)$. Let $\mathcal{N}_h$ denotes the set of the mesh nodes. Let ϕ_i be the basis of the linear finite element space S_h and u_h be the $\mathcal{C}^0$ finite element approximation of u on a partition of I. For an inner vertex $x_i \in \mathcal{N}_h$, $\mathcal{K}_i = I_i \cup I_{i+1}$ denotes its corresponding patch. For convenience, we denote

$$u_i = u_h(x_i), \qquad u'_{i-\frac{1}{2}} = u'_h(x_{i-\frac{1}{2}}).$$

From the definitions of simple averaging, geometry and harmonic averaging in Section 2, we see that all three methods results in some finite difference schemes involving values of u_h at x_{i-1}, x_i, x_{i+1}, the difference among the three weighted averaging techniques are the corresponding weights. In this way, we obtain values of the recovered derivatives at $x_1, x_2, \cdots, x_{N-1}$. If $u'(x_0)$ and $u'(x_N)$ are not provided by the problem, we simple define

$$(R_h u_h)(x_0) = \frac{(u_1 - u_0)}{h_1}, \qquad (R_h u_h)(x_N) = \frac{(u_N - u_{N-1})}{h_N}.$$

Finally, by linear interpolation, we recover a piecewise linear continuous derivative field, which is a better approximation of u' than u'_h. In the following, we will analysis the properties of these weighted averaging operators.

3.1 Properties of weighted averaging operators

Now, we give a theoretical analysis of the three weighted averaging recovery operators mentioned above. Let u_I be the interpolation of u on the finite element space S_h. Using Taylor expansion,

$$u(x_{i-1}) = u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2} u''(x_i) - \frac{h_i^3}{3!} u'''(x_i) + \mathcal{O}(h^4), \tag{3.3a}$$

$$u(x_{i+1}) = u(x_i) + h_{i+1} u'(x_i) + \frac{h_{i+1}^2}{2} u''(x_i) + \frac{h_{i+1}^3}{3!} u'''(x_i) + \mathcal{O}(h^4). \tag{3.3b}$$

By substituting the above equations into (2.1)-(2.3), and simplifying, we obtain

$$(S_h u_I)(x_i) = u'_i + \frac{h_{i+1} - h_i}{4} u''_i + \frac{h_i^2 + h_{i+1}^2}{12} u'''_i + \mathcal{O}(h^3), \tag{3.4a}$$

$$(G_h u_I)(x_i) = u'_i + \frac{h_{i+1} - h_i}{2} u''_i + \frac{h_i^2 - h_i h_{i+1} + h_{i+1}^2}{6} u'''_i + \mathcal{O}(h^3), \tag{3.4b}$$

$$(H_h u_I)(x_i) = u'_i + 0 u''_i + \frac{h_i h_{i+1}}{6} u'''_i + \mathcal{O}(h^3). \tag{3.4c}$$

If $h_i = h = h_{i+1}$, all three methods results in the same central difference scheme. However, when $h_i \neq h_{i+1}$, only harmonic averaging yields a second order finite difference scheme at x_i, and simple averaging is better than geometry averaging on the aspect of the coefficient of $u''(x_i)$.

Now, we state that harmonic averaging is the unique of weighted averaging methods which yields a second-order approximation of $u'(x_i)$ under any mesh. We consider the following problem: find the weight coefficients ω_i, $\omega_{i+1} > 0$, $\omega_i + \omega_{i+1} = 1$, such that

$$\begin{aligned}(R_h u)(x_i) =&\omega_i u'(x_{i-\frac{1}{2}}) + \omega_{i+1} u'(x_{i+\frac{1}{2}}) \\ =&\omega_i \cdot \frac{u(x_i) - u(x_{i-1})}{h_i} + \omega_{i+1} \cdot \frac{u(x_{i+1}) - u(x_i)}{h_{i+1}} = u'(x_i) + \mathcal{O}(h^2).\end{aligned}$$

By Taylor expansion (3.3a) and (3.3b), we obtain

$$\begin{aligned}(R_h u)(x_i) = (\omega_i + \omega_{i+1})u'(x_i) &+ \frac{h_{i+1}\omega_{i+1} - h_i\omega_i}{2} u''(x_i) \\ &+ \frac{h_i^2\omega_i + h_{i+1}^2\omega_{i+1}}{6} u'''(x_i) + \mathcal{O}(h^3).\end{aligned}$$

It requires that the weight coefficient to be satisfied

$$\left\{\begin{array}{l} \omega_i + \omega_{i+1} = 1, \\ h_{i+1}\omega_{i+1} - h_i\omega_i = 0, \\ \omega_i, \omega_{i+1} > 0, \end{array}\right. \Longrightarrow \left\{\begin{array}{l} \omega_i = \dfrac{h_{i+1}}{h_i + h_{i+1}} = \dfrac{1/h_i}{1/h_i + 1/h_{i+1}}, \\ \omega_{i+1} = \dfrac{h_i}{h_i + h_{i+1}} = \dfrac{1/h_{i+1}}{1/h_i + 1/h_{i+1}}. \end{array}\right.$$

This is just the weighted coefficient of harmonic averaging.

We summarize the result in the following proposition.

Proposition 3.1. *For the 1D problems, the harmonic averaging recovery operator is the optimal one, and also a superconvergence recovery operator under any mesh. All of the three weighted averaging methods yield the same result under uniform mesh.*

To get a better understanding of the recovery operator, we discuss an example in detail.

Example 3.1. Weighted averaging for linear element. Let the exact solution is a quadratic polynomial on the element patch $\mathcal{K}_i$. For simplicity of discussion, we assume that $u(x) = c_0 + c_1 x + c_2 x^2$, and $u'(x) = c_1 + 2c_2 x$. Following the procedure of simple averaging, geometry averaging, and harmonic averaging, we obtain

$$\begin{aligned}&(S_h u_I)(x_i) = c_1 + c_2 \frac{x_{i-1} + 2x_i + x_{i+1}}{2}, \\ &(G_h u_I)(x_i) = c_1 + c_2(x_{i-1} + x_{i+1}), \\ &(H_h u_I)(x_i) = c_1 + 2c_2 x_i,\end{aligned}$$

and

$$
\begin{aligned}
&(S_h u_I)(x_i) - u'(x_i) = c_2 \frac{x_{i-1} - 2x_i + x_{i+1}}{2}, \\
&(G_h u_I)(x_i) - u'(x_i) = c_2(x_{i-1} + x_{i+1} - 2x_i), \\
&(H_h u_I)(x_i) - u'(x_i) = 0.
\end{aligned}
$$

Let

$$
\begin{aligned}
e_s(x_i) &= |u'(x_i) - (S_h u_I)(x_i)|, \\
e_g(x_i) &= |u'(x_i) - (G_h u_I)(x_i)|, \\
e_h(x_i) &= |u'(x_i) - (H_h u_I)(x_i)|.
\end{aligned}
$$

Therefore, we have

$$
0 = e_h \le e_s \le e_g = |c_2(x_{i-1} - 2x_i + x_{i+1})|,
$$

and the equality holds if and only if the mesh is uniform.

As we can see, for the quadratic polynomial, the recovered derivative from harmonic averaging is the exact derivative $u'(x_i)$ at vertices. Then, for any solution, the harmonic averaging is a second-order approximation of u', while the simple averaging and geometry averaging are a first-order approximation except under the uniform mesh.

3.2 Theoretical results in 1D

By studying the resulting finite difference schemes of different recovery techniques, we have the following observations:

- For the above three weighted averaging methods, the harmonic averaging is the optimal one, and geometry averaging is the worst.
- Harmonic averaging results in superconvergence recovery operators under any mesh.

Based on the above observation, we analyze harmonic averaging here. There weak formulation of (3.1) is to find $u \in H_0^1(I)$ such that

$$
(au', v') + (bu, v) = (f, v), \quad \forall v \in H_0^1(I). \tag{3.5}
$$

By denoting the partition $\mathcal{T}_h = \{x_i\}_{i=0}^N$ and defining the finite element space

$$
S_h = \{v \in H^1(I), v|_{I_i} \in P_1(I_i)\}, \qquad S_h^0 = \{v \in H_0^1(I), v|_{I_i} \in P_1(I_i)\}.
$$

The finite element solution of (3.5) is to find $u_h \in S_h^0$, such that

$$
(au_h', v_h') + (bu_h, v_h) = (f, v_h), \quad \forall v_h \in S_h^0. \tag{3.6}
$$

The recovered derivative is a continuous piecewise linear polynomial (as u_h), $H_h u_h \in S_h$, which can be uniquely determined by its values $(H_h u_h)(x_i)$ at the vertices $x_i, i = 0, 1, \cdots, N$.

Subtracting (3.6) from (3.5) yields

$$(a(u' - u_h'), v') + (b(u - u_h), v) = 0, \qquad \forall v \in S_h^0. \tag{3.7}$$

Let $\tilde{u}_h \in S_h^0$ be the solution of

$$(u' - \tilde{u}_h', v') = 0, \qquad \forall v \in S_h^0. \tag{3.8}$$

Then we have the following "super-approximaion" result between u_h and $\tilde{u}_h$ (see [19]).

Lemma 3.1. *([19]) Let $u_h, \tilde{u}_h$ satisfy* (3.7), (3.8)*, respectively. Then there exists a constant C, independent of h and u, such that*

$$\|u_h' - \tilde{u}_h'\|_{0,\infty,I} \le Ch^2 \|u\|_{2,\infty,I}. \tag{3.9}$$

We introduce a simple case:

$$-u'' = f \quad \text{in } I = (0,1), \qquad u(0) = u(1) = 0,$$

or

$$(u', v') = (f, v), \quad \forall v \in H_0^1(I). \tag{3.10}$$

Note that the finite element solution of (3.10) satisfies (3.8).

In the following, we will prove superconvergence property of the recovered derivative. The basis functions of S_h^0 are the standard Lagrange basis functions

$$S_h^0 = span\{N_i(x), i = 1, 2, \cdots, N-1\}.$$

Here,

$$N_i(x) = \begin{cases} 1 + (x - x_i)/h_i, & x \in I_i, \\ 1 + (x_i - x)/h_{i+1}, & x \in I_{i+1}, \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 3.2. *Let u be the solution of* (3.10)*, and u_h be its correspond finite element approximation on S_h^0. Assume that u is a quadratic polynomial. Then $H_h u_h = u'$ for harmonic averaging.*

Proof. Consider the case on an patch $J_i = (x_{i-2}, x_{i+1})$. First, we are able to express explicitly the finite element solution of (3.10) on I_i as

$$u_h(x) = u(x_{i-1}) N_{i-1}(x) + u(x_i) N_i(x). \tag{3.11}$$

Then we have, on I_i,

$$u_h'(x) = \frac{u(x_i) - u(x_{i-1})}{h_i}. \tag{3.12}$$

As $u \in P_2(J_i)$, we have $u' \in P_1(J_i)$, and

$$u(x) = c_0 + c_1 x + c_2 x^2, u'(x) = c_1 + 2c_2 x = u'(x_{i-1})N_{i-1}(x) + u'(x_i)N_i(x), \quad x \in I_i. \tag{3.13}$$

Following the procedure of harmonic averaging, we obtain that

$$\begin{aligned}
(H_h u_h)(x_{i-1}) &= \frac{h_i}{h_i + h_{i-1}} \cdot \frac{u(x_{i-1}) - u(x_{i-2})}{h_{i-1}} + \frac{h_{i-1}}{h_i + h_{i-1}} \cdot \frac{u(x_i) - u(x_{i-1})}{h_i} \\
&= c_1 + 2c_2 x_{i-1} = u'(x_{i-1}), \\
(H_h u_h)(x_i) &= \frac{h_{i+1}}{h_i + h_{i+1}} \cdot \frac{u(x_i) - u(x_{i-1})}{h_i} + \frac{h_i}{h_i + h_{i+1}} \cdot \frac{u(x_{i+1}) - u(x_i)}{h_{i+1}} \\
&= c_1 + 2c_2 x_i = u'(x_i).
\end{aligned}$$

Then, on I_i,

$$(H_h u_h)(x) = (H_h u_h)(x_{i-1})N_{i-1}(x) + (H_h u_h)(x_i)N_i(x) = u'(x).$$

Applying the same argument on I_{i-1}, I_{i+1}, we have

$$(H_h u_h)(x) = u'(x), \quad x \in J_i.$$

Using the same argument on all patch J_i, we then complete the proof. $\square$

Applying the Bramble-Hilbert lemma and Lemma 3.1, a direct consequence of Theorem 3.2 is the following suerconvergence property.

Theorem 3.3. *Let u be the solution of* (3.5) *and u_h be its correspond finite element approximation on S_h^0. Then there exists a constant C, independent of h and u, such that for harmonic averaging, at an interior node x_i*

$$|u'(x_i) - (H_h u_h)(x_i)| \le Ch^2(|u|_{3,\infty,J_i} + \|u\|_{2,\infty,I}). \tag{3.14}$$

Proof. There exists a function $\tilde{f}$ which satisfies

$$(u', v') = (\tilde{f}, v), \quad \forall v \in H_0^1(I).$$

Assume $\tilde{u}_h \in S_h^0$ is the solution of

$$(u' - \tilde{u}_h', v_h') = 0, \quad \forall v_h \in S_h^0.$$

Following Theorem 3.2 and the standard argument by applying the Bramble-Hilbert lemma, we have

$$\begin{aligned}
|u'(x_i) - (H_h \tilde{u}_h)(x_i)| &= |u'(x_i) - (H_h u_I)(x_i)| = |u'(x_i) - (H_h u)(x_i)| \\
&\le Ch^2 |u|_{3,\infty,J_i}.
\end{aligned} \tag{3.15}$$

Notice that

$$|u'(x_i) - (H_h u_h)(x_i)| \le |u'(x_i) - (H_h \tilde{u}_h)(x_i)| + |(H_h \tilde{u}_h)(x_i) - (H_h u_h)(x_i)|.$$

From Lemma 3.1 and (3.15), we obtain that

$$|u'(x_i) - (H_h u_h)(x_i)| \le Ch^2(|u|_{3,\infty,J_i} + \|u\|_{2,\infty,I}).$$

Then we complete the proof. $\square$

Now, the superconvergence property is available for the harmonic averaging, while it is not for simple averaging and geometry averaging. However we should point out that the generalization of the result to the higher-dimensional tensor product case is not straightforward.

4 Weighted averaging gradient recovery for rectangular element

Let $\Omega \in \mathbb{R}^2$ be a bounded rectangular domain, we consider the following boundary value problem:

$$\begin{cases} -\nabla \cdot (A\nabla u) + cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume that all the coefficient functions are sufficiently smooth in our analysis and A is a 2×2 symmetric positive definite matrix and $c \ge 0$.

In weak form, this problem reads: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{4.2}$$

where

$$a(u,v) = \int_\Omega [A\nabla u \nabla v + cuv] dx,$$

and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. We assume that the bilinear operator $a(\cdot,\cdot)$ is continuous and $H_0^1(\Omega)$-elliptic, i.e., there exists a constant $a_0 > 0$ such that

$$A(v,v) \ge a_0 \|v\|_{1,\Omega}, \quad \forall v \in H_0^1(\Omega).$$

Under these assumptions, the variational problem in (4.2) has a unique weak solution $u \in H_0^1(\Omega)$.

Let $\mathcal{T}_h$ be a rectangular partition of Ω and S_h be a bilinear finite element space over $\mathcal{T}_h$ defined by

$$S_h = \{v \in H^1(\Omega) : v \in Q_1(\tau),\ \forall \tau \in \mathcal{T}_h\}, \quad S_h^0 = \{v \in H_0^1(\Omega) : v \in Q_1(\tau),\ \forall \tau \in \mathcal{T}_h\},$$

where $Q_k(D)$ denotes the set of polynomials defined in $D \in \mathbb{R}^2$ with degree no more than k in each variable. The finite element solution of (4.2) is to find $u_h \in S_h^0$ such that

$$a(u_h, v) = (f, v), \quad \forall v \in S_h^0. \tag{4.3}$$

In this section, we consider the simple averaging, geometry averaging and harmonic averaging methods which are defined by (2.4)-(2.6) in Section 2. We firstly result the three weighted averaging methods in the finite difference schemes, and discuss some examples in details. Then we analysis the property of our new harmonic averaging recovery operator in the end.

4.1 Weighted averaging schemes and examples

As in one dimension, we consider three weighted averaging methods of gradient recovery for rectangular element. The parameters associated with the considered node z are displayed in Fig. 1(a). There are some relationships among the weights:

$$S_1 = h_2k_1, \quad S_2 = h_1k_1, \quad S_3 = h_1k_2, \quad S_4 = h_2k_2, \quad S = \sum_{i=1}^{4} S_i = (h_1+h_2)(k_1+k_2),$$
$$\sum_{i=1}^{4}\frac{1}{S_i} = \frac{S}{h_1h_2k_1k_2}, \quad \frac{1/S_1}{\sum_{j=1}^4 1/S_j} = \frac{S_3}{S}, \quad \frac{1/S_2}{\sum_{j=1}^4 1/S_j} = \frac{S_4}{S}, \quad \frac{1/S_3}{\sum_{j=1}^4 1/S_j} = \frac{S_1}{S}, \quad \frac{1/S_4}{\sum_{j=1}^4 1/S_j} = \frac{S_2}{S}.$$

In the following, we result the simple averaging, geometry averaging, and harmonic averaging methods in the finite difference schemes and give some examples to illustrate the idea. With no confusion, we abbreviate u_I to u for convenience.

Let $R_h^x v$ and $R_h^y v$ ($R_h = S_h, G_h, H_h$) denote the recovered derivative of v in x and y direction, respectively. Using Taylor series, (2.4), (2.5), (2.6), and simplifying, we obtain

$$\begin{aligned}
(S_hu_I)(z) &= \sum_{i=1}^{4}\frac{1}{4}\nabla u_I(z)|_{\tau_i} = \frac{1}{2}\begin{pmatrix} \dfrac{u(z_1)-u(z)}{h_2} + \dfrac{u(z)-u(z_5)}{h_1} \\ \dfrac{u(z_3)-u(z)}{k_1} + \dfrac{u(z)-u(z_7)}{k_2} \end{pmatrix} \\
&= \begin{pmatrix} u_x(z) + \dfrac{h_2-h_1}{4}u_{xx}(z) + \dfrac{h_1^2+h_2^2}{12}u_{xxx}(z) + \mathcal{O}(h^3) \\ u_y(z) + \dfrac{k_1-k_2}{4}u_{yy}(z) + \dfrac{k_1^2+k_2^2}{12}u_{yyy}(z) + \mathcal{O}(h^3) \end{pmatrix},
\end{aligned} \tag{4.4a}$$

$$\begin{aligned}
(G_hu_I)(z) &= \sum_{i=1}^{4}\frac{S_i}{\sum_{j=1}^4 S_j}\nabla u_I(z)|_{\tau_i} = \begin{pmatrix} \dfrac{u(z_1)-u(z_5)}{h_1+h_2} \\ \dfrac{u(z_3)-u(z_7)}{k_1+k_2} \end{pmatrix} \\
&= \begin{pmatrix} u_x(z) + \dfrac{h_2-h_1}{2}u_{xx}(z) + \dfrac{h_1^2-h_1h_2+h_2^2}{6}u_{xxx}(z) + \mathcal{O}(h^3) \\ u_y(z) + \dfrac{k_1-k_2}{2}u_{yy}(z) + \dfrac{k_1^2-k_1k_2+k_2^2}{6}u_{yyy}(z) + \mathcal{O}(h^3) \end{pmatrix},
\end{aligned} \tag{4.4b}$$

$$
\begin{aligned}
(H_h u_I)(z) &= \sum_{i=1}^{4} \frac{1/S_i}{\sum_{j=1}^{4} 1/S_j} \nabla u_I(z)|_{\tau_i} = \begin{pmatrix} \dfrac{h_1}{h_1+h_2}\dfrac{u(z_1)-u(z)}{h_2} + \dfrac{h_2}{h_1+h_2}\dfrac{u(z)-u(z_5)}{h_1} \\ \dfrac{k_2}{k_1+k_2}\dfrac{u(z_3)-u(z)}{k_1} + \dfrac{k_1}{k_1+k_2}\dfrac{u(z)-u(z_7)}{k_2} \end{pmatrix} \\
&= \begin{pmatrix} u_x(z) + \dfrac{h_1 h_2}{6} u_{xxx}(z) + \mathcal{O}(h^3) \\ u_y(z) + \dfrac{k_1 k_2}{6} u_{yyy}(z) + \mathcal{O}(h^3) \end{pmatrix},
\end{aligned} \tag{4.4c}
$$

for choosing z as the sampling point. If the elements central points are chosen as the sampling points, then

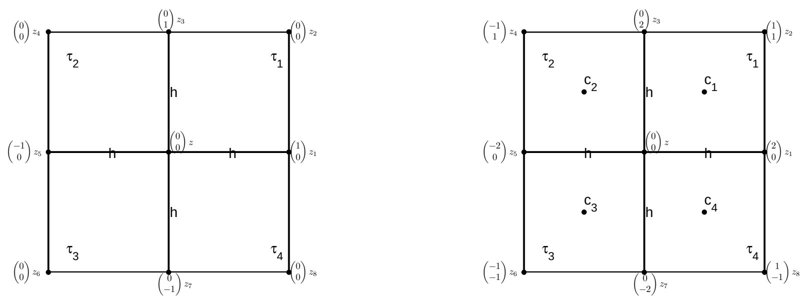
$$
\begin{aligned}
(S_h u_I)(z) &= \sum_{i=1}^{4} \frac{1}{4} \nabla u_I(c_i) = \begin{pmatrix} u_x(z) + \dfrac{h_2-h_1}{4} u_{xx}(z) + \dfrac{k_1-k_2}{4} u_{xy}(z) \\ u_y(z) + \dfrac{k_1-k_2}{4} u_{yy}(z) + \dfrac{h_2-h_1}{4} u_{xy}(z) \end{pmatrix} \\
&\quad + \begin{pmatrix} \dfrac{h_1^2+h_2^2}{12} u_{xxx}(z) + \dfrac{(h_2-h_1)(k_1-k_2)}{16} u_{xxy}(z) + \dfrac{k_1^2+k_2^2}{8} u_{xyy}(z) + \mathcal{O}(h^3) \\ \dfrac{k_1^2+k_2^2}{12} u_{yyy}(z) + \dfrac{h_1^2+h_2^2}{8} u_{xxy}(z) + \dfrac{(h_2-h_1)(k_1-k_2)}{16} u_{xyy}(z) + \mathcal{O}(h^3) \end{pmatrix},
\end{aligned} \tag{4.5a}
$$

$$
\begin{aligned}
(G_h u_I)(z) &= \sum_{i=1}^{4} \frac{S_i}{\sum_{j=1}^{4} S_j} \nabla u_I(c_i) = \begin{pmatrix} u_x(z) + \dfrac{h_2-h_1}{2} u_{xx}(z) + \dfrac{k_1-k_2}{2} u_{xy}(z) \\ u_y(z) + \dfrac{k_1-k_2}{2} u_{yy}(z) + \dfrac{h_2-h_1}{2} u_{xy}(z) \end{pmatrix} \\
&\quad + \begin{pmatrix} \dfrac{h_1^2-h_1h_2+h_2^2}{6} u_{xxx}(z) + \dfrac{(h_2-h_1)(k_1-k_2)}{4} u_{xxy}(z) + \dfrac{k_1^2-k_1k_2+k_2^2}{4} u_{xyy}(z) + \mathcal{O}(h^3) \\ \dfrac{k_1^2-k_1k_2+k_2^2}{6} u_{yyy}(z) + \dfrac{h_1^2-h_1h_2+h_2^2}{4} u_{xxy}(z) + \dfrac{(h_2-h_1)(k_1-k_2)}{4} u_{xyy}(z) + \mathcal{O}(h^3) \end{pmatrix},
\end{aligned} \tag{4.5b}
$$

$$
(H_h u_I)(z) = \sum_{i=1}^{4} \frac{1/S_i}{\sum_{j=1}^{4} 1/S_j} \nabla u_I(c_i) = \begin{pmatrix} u_x(z) + \dfrac{h_1h_2}{6} u_{xxx}(z) + \dfrac{k_1k_2}{4} u_{xyy}(z) + \mathcal{O}(h^3) \\ u_y(z) + \dfrac{k_1k_2}{6} u_{yyy}(z) + \dfrac{h_1h_2}{4} u_{xxy}(z) + \mathcal{O}(h^3) \end{pmatrix}. \tag{4.5c}
$$

To get a better understanding of the recovery operator, we give some detailed examples.

Example 4.1. Weighted averaging on uniform rectangular mesh. An element patch with horizontal and vertical edge length h is displayed in Fig. 2. Note that the weights in the three averaging method are the same value $1/4$, the resulting finite difference operator are shown in Fig. 2. Both of them are second-order finite difference schemes.

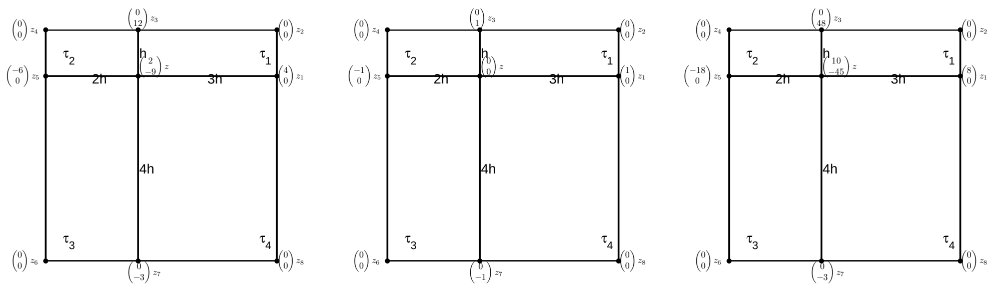


(a) Denominator $2h$, sampling point z (b) Denominator $8h$, sampling point c_i

Figure 2: Example 4.1, weighted averaging on uniform mesh.

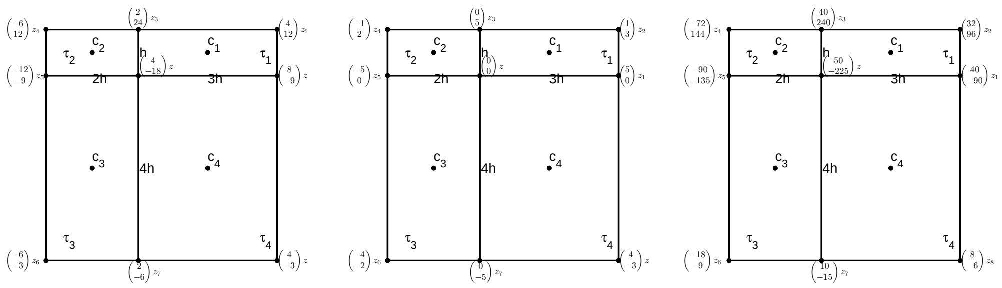
One can see that, under the uniform mesh, all the three averaging methods result in the same finite difference schemes.

Example 4.2. Weighted averaging on nonuniform rectangular mesh. An element patch with the parameters is displayed in Fig. 4. From (2.4)-(2.6), the resulting finite difference operator are reported in Fig. 3-4. By Taylor expansion, it is straightforward to verify that simple averaging and geometry averaging are first order difference schemes for both derivatives, while harmonic averaging is a second order difference schemes.



(a) Denominator $24h$ (b) Denominator $5h$ (c) Denominator $60h$

Figure 3: Example 4.2, weights, sampling point z. (a): Simple averaging, (b): Geometry averaging, (c): Harmonic averaging.



(a) Denominator $90h$ (b) Denominator $50h$ (c) Denominator $600h$

Figure 4: Example 4.2, weights, sampling point c_i. (a): Simple averaging, (b): Geometry averaging, (c): Harmonic averaging.

Some remark are listed below:

- Under the uniform rectangular mesh, all the three weighted averaging recovery techniques produce the same recovery operator at the patch center, as seen in Fig. 2. By Taylor expansion, it is straightforward to verify that this is a second order difference scheme for both derivatives. Indeed, numerical tests confirm that recovered gradient convergences at a rate $\mathcal{O}(h^2)$ which is one order higher than the rate of the original piecewise linear function gradient. Therefore, in this case, we have a superconvergence recovery for all three averaging methods.
- Under the nonuniform rectangular mesh, simple averaging, geometry averaging, and harmonic averaging produce the different recovery schemes, as seen in Figs. 3-4. Taylor expansion reveals that harmonic averaging results in a second order finite difference operator at the

assemble point while the other two produce first order finite difference schemes. Numerical results confirm that the harmonic averging scheme provides a superconvergence recovery at a rate $\mathcal{O}(h^2)$, while the simple averaging and geometry averging methods are linear recovery operators.

- For the three weighted averaging methods, we use two strategies to choose sampling points, one is the assemble point itself, the other is the element central points in patch. It is clear that, for the interpolation of u, the error $|\nabla u(z) - (R_h u_I)(z)|$ of using the assemble itself as the sampling point is better than using the element central points. While for the finite element solution, sometimes using the element central points can obtain a better results for the central point is the superconvergent point for the bilinear element. Both of them produce the same order difference scheme, but using the assemble point is much simpler than using the element central points. So, from now on, we only use the assemble point as the sample points.

4.2 Properties of the gradient recovery operator

In the following, we show that the weights of harmonic averaging is sufficient condition to yield a superconvergence recovery operator under any rectangular mesh, but it is not necessary condition. For the element patch $\mathcal{K}_z$, as shown in Fig. 1(a), find the weights $\omega_i > 0$, $\sum_{i=1}^4 \omega_i = 1$ such that

$$(R_h u_I)(z) = \sum_{i=1}^{4} \omega_i \nabla u_I(z)|_{\tau_i} = \nabla u(z) + \mathcal{O}(h^2).$$

From the definition of the weighted averaging recovery operator

$$(R_h u_I)(z) = \sum_{i=1}^{4} \omega_i \nabla u_I(z)|_{\tau_i} = \begin{pmatrix} (\omega_1+\omega_4)\dfrac{u(z_1)-u(z)}{h_2} + (\omega_2+\omega_3)\dfrac{u(z)-u(z_5)}{h_1} \\ (\omega_1+\omega_2)\dfrac{u(z_3)-u(z)}{k_1} + (\omega_3+\omega_4)\dfrac{u(z)-u(z_7)}{k_2} \end{pmatrix}.$$

By Taylor expansion, we have

$$(R_h u_I)(z) = \begin{pmatrix} \sum\limits_{i=1}^{4} \omega_i u_x(z) + \dfrac{1}{2}(h_2(\omega_1+\omega_4) - h_1(\omega_2+\omega_3))u_{xx}(z) + \mathcal{O}(h^2) \\ \sum\limits_{i=1}^{4} \omega_i u_y(z) + \dfrac{1}{2}(k_1(\omega_1+\omega_2) - k_2(\omega_3+\omega_4))u_{yy}(z) + \mathcal{O}(h^2) \end{pmatrix}.$$

To make this is a second order approximation to $\nabla u(z)$, the weights should satisfy the following relationship

$$\begin{cases} \sum\limits_{i=1}^{4} \omega_i = 1, \qquad \omega_i > 0, \\ h_2(\omega_1+\omega_4) - h_1(\omega_2+\omega_3) = 0, \\ k_1(\omega_1+\omega_2) - k_2(\omega_3+\omega_4) = 0, \end{cases} \Longrightarrow \begin{cases} \omega_1+\omega_2 = \dfrac{k_2}{k_1+k_2}, \quad \omega_2+\omega_3 = \dfrac{h_2}{h_1+h_2}, \\ \omega_3+\omega_4 = \dfrac{k_1}{k_1+k_2}, \quad \omega_1+\omega_4 = \dfrac{h_1}{h_1+h_2}. \end{cases}$$

The above linear equations are solvable, but not unique. It is obviously that the weights in harmonic averaging satisfy the above requirements, while simple averaging and geometry averaging are not.

Theorem 4.1. *Let $\mathcal{K}_z$ be the node patch, see Fig. 1(a). Assume u is a bi-quadratic polynomial on $\mathcal{K}_z$, then $(H_h u_I)(z) = \nabla u(z)$ for harmonic averaging which select the assemble point as the sampling point.*

Proof. Firstly, we see that

$$Q_2(\mathcal{K}_z) = span\{\phi_{i,j}(x,y) = x^i y^j,\ \ 0 \le i,j \le 2\}.$$

When $u \in Q_2(\mathcal{K}_z)$, we have

$$u = \sum_{i,j=0}^{2} c_{ij}\phi_{i,j}(x,y).$$

Now, we turn to verify that

$$(H_h\phi_{i,j})(z) = \nabla\phi_{i,j}(z), \quad 0 \le i,j \le 2.$$

Take $\phi_{2,2}(x,y) = x^2y^2$ as an example, following the proceeding of harmonic averaging, let $z = (x_0, y_0)$, $z_i = (x_i, y_i)$, $1 \le i \le 8$, we have

$$(H_h\phi_{2,2})(z) = \begin{pmatrix} \dfrac{x_0 - x_5}{x_1 - x_5}\dfrac{x_1^2y_1^2 - x_0^2y_0^2}{x_1 - x_0} + \dfrac{x_1 - x_0}{x_1 - x_5}\dfrac{x_0^2y_0^2 - x_5^2y_5^2}{x_0 - x_5} \\ \dfrac{y_0 - y_7}{y_3 - y_7}\dfrac{x_3^2y_3^2 - x_0^2y_0^2}{y_3 - x_0} + \dfrac{y_3 - y_0}{y_3 - y_7}\dfrac{x_0^2y_0^2 - x_7^2y_7^2}{y_0 - y_7} \end{pmatrix} = \begin{pmatrix} 2x_0y_0^2 \\ 2x_0^2y_0 \end{pmatrix} = \nabla\phi_{2,2}(z).$$

Similarly, we can obtain $(H_h\phi_{i,j})(z) = \nabla\phi_{i,j}(z)$ for all $0 \le i,j \le 2$. Then

$$(H_h u_I)(z) = \sum_{i,j=0}^{2} c_{i,j}(H_h\phi_{i,j})(z) = \sum_{i,j=0}^{2} c_{i,j}\nabla\phi_{i,j}(z) = \nabla u(z).$$

Thus, the theorem is proved. $\square$

We can see that the harmonic averaging operator have the polynomial preserving property in the sense of $(H_h u)(z) = \nabla u(z)$ for $u \in Q_2(\mathcal{K}_z)$. The polynomial preserving recovery method [16, 23] fits a higher order polynomial for the function value of the solution by local least-squares fitting, and the recovered gradient is obtained by taking the gradient of the recovered polynomial. When $u \in P_2(\mathcal{K}_z)$, the least-squares fitting of a polynomial of degree 2 will reproduce u, therefore $G_h u(z) = \nabla u(z)$ (G_h-the PPR operator).

4.3 Theoretical results for rectangular element

Let u_I be a bi-linear interpolation of u on the rectangular mesh $\mathcal{T}_h$, we have the following superconvergent results.

Lemma 4.1. *(see [7]) Let Ω be a bounded rectangular domain, and $\mathcal{T}_h$ be a quasi-uniform rectangular partition. And $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$ is the solution of* (4.2). *Then we have the following superconvergent result between u and u_I*

$$\max_{\Omega} |\nabla(u_h - u_I)| \le ch^2|\ln h|\|u\|_{3,\infty,\Omega}. \tag{4.6}$$

Applying the Bramble-Hilbert lemma and Lemma 4.1, we have the following suer-convergence property.

Theorem 4.2. *Let u be the solution of* (4.2)*, and u_h be its corresponding finite element approximation on S_h^0. Assume the conditions on Lemma 4.1 hold. Then there exists a constant C, independent of h and u, such that for harmonic averaging, at an interior node z*

$$|\nabla u(z) - (H_h u_h)(z)| \leq Ch^2 |\ln h| (\|u\|_{3,\infty,\Omega}). \tag{4.7}$$

Proof. Notice that

$$|\nabla u(z) - (H_h u_h)(z)| \leq |\nabla u(z) - (H_h u_I)(z)| + |(H_h u_I)(z) - (H_h u_h)(z)|.$$

Similar as in proof of Theorem 3.3, (4.7) follows from Lemma 4.1, Theorem 4.1 and the standard argument by applying the Bramble-Hilbert lemma. □

5 Weighted averaging gradient recovery for triangular element

In this section, we present a counter example to state that there does not exist a weighted averaging method for gradient recovery under any triangular mesh such that

$$|(R_h u_I)(z) - \nabla u(z)| = \mathcal{O}(h^2).$$

We provide some numerical examples in the performance of these new weighted averaging methods together with simple averaging and geometry averaging in Section 6.

Example 5.1. As the procedure in rectangular element, we want to find a similar weighted averaging of gradient recovery for linear triangular element that the recovered gradient converges at a rate $\mathcal{O}(h^2)$ for any triangular mesh. In the following, a counterexample is given to illustrate that the weights are not exist.

An chevron element patch with horizontal and vertical edge length h is displayed in Fig. 5. We find the weights $\omega_i, i = 1, \cdots, 6$, $\sum_{i=1}^6 \omega_i = 1$ such that the recovered gradient of weighted averaging $(R_h u_I)(z) = \sum_{i=1}^6 \omega_i \nabla u_I|_{\tau_i}$ satisfy

$$\nabla u(z) - (R_h u_I)(z) = \nabla u(z) - \sum_{i=1}^{6} \omega_i \nabla u_I|_{\tau_i} = \mathcal{O}(h^2).$$

Noting that

$$(R_h u_I)(z) = \begin{pmatrix} (\omega_1+\omega_6)\dfrac{u(z_1)-u(z)}{h} + (\omega_2+\omega_3)\dfrac{u(z)-u(z_3)}{h} + \omega_4\dfrac{u(z_5)-u(z_4)}{h} + \omega_5\dfrac{u(z_6)-u(z_5)}{h} \\ (\omega_1+\omega_2)\dfrac{u(z_2)-u(z)}{h} + \omega_3\dfrac{u(z_3)-u(z_4)}{h} + (\omega_4+\omega_5)\dfrac{u(z)-u(z_5)}{h} + \omega_6\dfrac{u(z_1)-u(z_6)}{h} \end{pmatrix}.$$

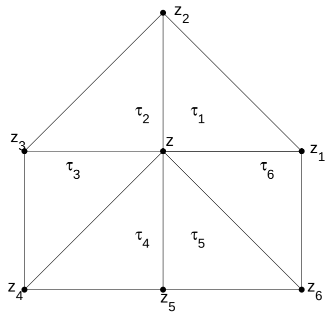


Figure 5: Parameters associated with the node z.

By Taylor expansion, we have that

$$(R_h u_I)(z) = \begin{pmatrix} \sum\limits_{i=1}^{6} \omega_i u_x(z) + \dfrac{h}{2}(\omega_1+\omega_5+\omega_6-\omega_2-\omega_3-\omega_4)u_{xx}(z) - h(\omega_4+\omega_5)u_{xy}(z) + \mathcal{O}(h^2) \\ \sum\limits_{i=1}^{6} \omega_i u_y(z) + h(\omega_6-\omega_3)u_{xy}(z) + \dfrac{h}{2}(\omega_1+\omega_2-\omega_3-\omega_4-\omega_5-\omega_6)u_{yy}(z) + \mathcal{O}(h^2) \end{pmatrix}.$$

It is required that the weights satisfy the following system

$$\begin{cases} \sum\limits_{i=1}^{6} \omega_i = 1, \\ \omega_i > 0, \\ \omega_4+\omega_5 = 0, \\ \omega_6-\omega_3 = 0, \\ \omega_1+\omega_5+\omega_6-\omega_2-\omega_3-\omega_4 = 0, \\ \omega_1+\omega_2-\omega_3-\omega_4-\omega_5-\omega_6 = 0, \end{cases} \Longrightarrow \begin{cases} \omega_i > 0, \\ \omega_4+\omega_5 = 0, \\ \omega_6-\omega_3 = 0, \\ \omega_1+\omega_2+2\omega_3 = 1, \\ \omega_1-\omega_2-2\omega_4 = 0, \\ \omega_1+\omega_2-2\omega_3 = 0. \end{cases} \tag{5.1}$$

It is obviously that the above system is unsolvable. Then there does not exist a weights of weighted averaging such that recovered gradient converges at a rate $\mathcal{O}(h^2)$ for triangular mesh with chevron pattern, even for a general triangular mesh.

If the weights can take as zero, then $\omega_1 = \omega_2 = \omega_3 = \omega_6 = 1/4$, $\omega_4 = \omega_5 = 0$ is a solutions of the above equations in (5.1). What's more, for the chevron pattern, if we using the different weights in weighted averaging method for gradient recovery each direction, a possible choose for weights is

$$\begin{aligned} (R_h^x u_I)(z) &= \frac{1}{4}\frac{\partial u_I}{\partial x}\Big|_{\tau_1} + \frac{1}{4}\frac{\partial u_I}{\partial x}\Big|_{\tau_2} + \frac{1}{4}\frac{\partial u_I}{\partial x}\Big|_{\tau_3} + \frac{1}{4}\frac{\partial u_I}{\partial x}\Big|_{\tau_6}, \\ (R_h^y u_I)(z) &= \frac{1}{4}\frac{\partial u_I}{\partial y}\Big|_{\tau_1} + \frac{1}{4}\frac{\partial u_I}{\partial y}\Big|_{\tau_2} + \frac{1}{4}\frac{\partial u_I}{\partial y}\Big|_{\tau_4} + \frac{1}{4}\frac{\partial u_I}{\partial y}\Big|_{\tau_5}. \end{aligned}$$

For the triangular element, though we can not obtain the same good results as in rectangular element, but under some certain conditions on the mesh, these five weighted averaging methods also yield the superconvergent approximation for gradient. We will provide numerical evidence to show such phenomena in the next section.

6 Numerical examples

The test example is the Possion equation with the Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (-1,1)$ or $\Omega = (0,1) \times (0,1)$. One can observe the weighted averaging recovery methods' (Geometry Averaging-G_h, Simple Averaging-S_h, Harmonic Averaging-H_h, Angle averaging-A_h, Distant averaging-D_h) performance on difference type of meshes. Let $\mathcal{N}_h$ denotes the set of inner mesh nodes and N is the number of inner mesh nodes, we use the error notation

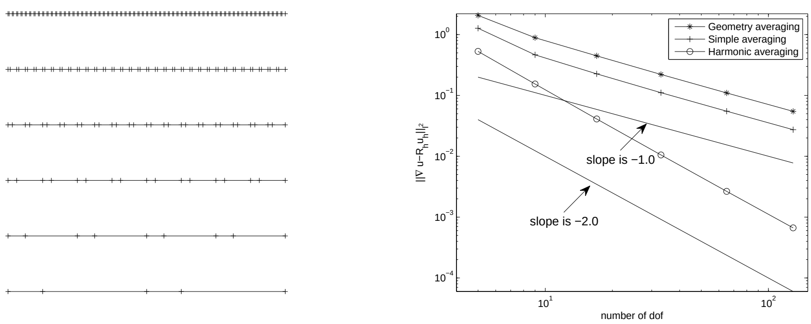
$$e_{R_h} = \|\nabla u - R_h u_h\|_{l^2} = \Big(\frac{1}{N}\sum_{z\in\mathcal{N}_h} \nabla u(z) - (R_h u_h)(z)\Big)^{\frac{1}{2}},$$

here R_h denote the corresponding weighted averaging recovery operator. To compare clearer, we only calculate the error on the inner points.

6.1 Test case 1

Our first example is on the interval $(-1,1)$ with a exact solution $u = \sin(\pi x)$. This example is carried on the nonuniform mesh, the results are displayed in Fig. 6, which compares the performance of the simple averaging recovery, geometry averaging recovery and our new harmonic averaging recovery. The numerical results show that, on the nonuniform meshes, the harmonic averaging recovery method has a second order convergent rate, i.e., a superconvergence property, while the simple averaging and geometry averaging recovery methods only have a first order convergent rate.

In the following Test case 2, we investigate the performance of simple averaging, geometry averaging and harmonic averaging recovery methods under rectangular meshes. Computation is performed on uniform mesh and nonuniform mesh for finite element solution u_h and we test the two strategies on the sampling points.

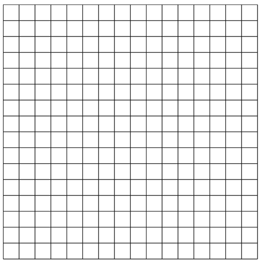


(a) Nonuniform meshes (b) Error

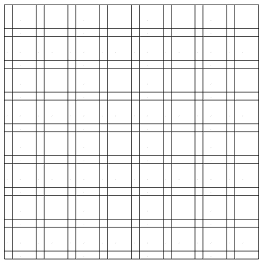
Figure 6: Test case 1, results on nonuniform meshes.

6.2 Test case 2

Our second example is on the domain $(0,1) \times (0,1)$ with a exact solution $u = \sin(\pi x)\sin(\pi y)$, the meshes are displayed in Fig. 7. The recovered error $\|\nabla u - R_h u_h\|_{l^2}$ for finite element solution u_h are listed in Table 1. Fig. 8 shows the corresponding results. We see clearly that the harmonic averaging yields superconvergence for rectangular element on both uniform mesh and nonuniform mesh, while the geometry averaging and simple averaging do not have on nonuniform mesh.

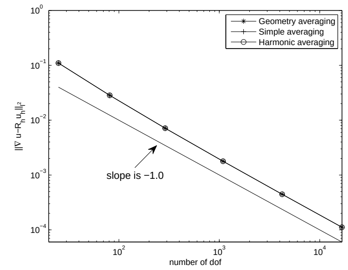


(a) Uniform mesh

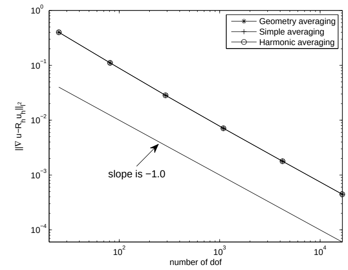


(b) Nonuniform mesh

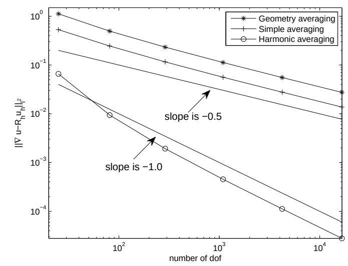
Figure 7: Rectangular meshes.



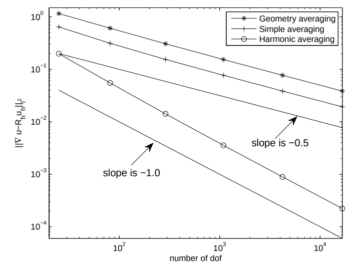
(a)



(b)



(c)



(d)

Figure 8: Test case 2, rectangular mesh, results for finite element solution u_h; (a) Uniform mesh, sample point: node; (b) Uniform mesh, sample point: element central points; (c) Nonuniform mesh, sample point: node; (d) Nonuniform mesh, sample point: element central points.

In the following, we present numerical investigation in the performance of the five weighted averaging methods for triangular element, the exact solution is selected as $u = e^{x^2+y^2}$. One can find the five weighted averaging recovery methods' (Geometry

Table 1: Test case 2, results for the finite element solution u_h.

Solution: $u=\sin(\pi x)\sin(\pi y)$, sampling point: z.						
	Uniform mesh, Fig. 7, (a)			Nonuniform Mesh, Fig. 7, (b)		
Dof	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$
25	0.109998	0.109998	0.109998	1.12102	0.530958	0.0658275
81	0.0284023	0.0284023	0.0284023	0.495821	0.245019	0.00941385
289	0.00713027	0.00713027	0.00713027	0.232393	0.115867	0.00192238
1089	0.00178389	0.00178389	0.00178389	0.112538	0.056229	0.000454483
4225	0.00044604	0.00044604	0.00044604	0.0553885	0.0276893	0.000112101
16641	0.000111542	0.000111542	0.000111542	0.0274772	0.013738	2.7934e-5
Solution: $u=\sin(\pi x)\sin(\pi y)$, sampling point: c_i.						
	Uniform mesh, Fig. 7, (a)			Nonuniform Mesh, Fig. 7, (b)		
Dof	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$
25	0.400606	0.400606	0.400606	1.16159	0.643609	0.198901
81	0.111003	0.111003	0.111003	0.610169	0.313983	0.0551299
289	0.0283565	0.0283565	0.0283565	0.307588	0.154921	0.0141543
1089	0.00712523	0.00712523	0.00712523	0.154104	0.0771918	0.00356118
4225	0.00178351	0.00178351	0.00178351	0.0770968	0.0385658	0.00089156
16641	0.000446029	0.000446029	0.000446029	0.0385536	0.019276	0.000222973

Average-G_h, Simple Average-S_h, Angle Average-A_h, Distant Average-D_h Harmonic Average-H_h) performance on different types of meshes.

6.3 Test case 3

We consider the five weighted averaging recovery techniques on the uniform mesh of four patterns, as seen in Fig. 9. Corresponding results are given in Table 2. The

Table 2: Test case 3, results for the finite element solution u_h.

Solution: $u=e^{x^2+y^2}$, uniform meshes.					
Regular pattern, Fig. 9, (a) and (e)					
Dof	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$	$\|\nabla u-A_hu_h\|_{l^2}$	$\|\nabla u-D_hu_h\|_{l^2}$
121	0.123524	0.123524	0.123524	0.109522	0.114423
441	0.0339251	0.0339251	0.0339251	0.030092	0.03143343
1681	0.0089016	0.0089016	0.0089016	0.00789611	0.00824797
6561	0.00228057	0.00228057	0.00228057	0.00202294	0.00211309
25921	0.000577206	0.000577206	0.000577206	0.000511992	0.0000534813
Chevron pattern, Fig. 9, (b) and (f)					
Dof	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$	$\|\nabla u-A_hu_h\|_{l^2}$	$\|\nabla u-D_hu_h\|_{l^2}$
121	0.205945	0.205945	0.205945	0.12118	0.138413
441	0.110379	0.110379	0.110379	0.0596393	0.0712069
1681	0.0573133	0.0573133	0.0573133	0.0302746	0.0366223
6561	0.0292302	0.0292302	0.0292302	0.0153722	0.0186517
25921	0.0147646	0.0147646	0.0147646	0.00776224	0.00942322
Criss-cross pattern, Fig. 9, (c) and (g)					
Dof	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$	$\|\nabla u-A_hu_h\|_{l^2}$	$\|\nabla u-D_hu_h\|_{l^2}$
61	0.175492	0.175492	0.175492	0.175492	0.175492
221	0.0494963	0.0494963	0.0494963	0.0494963	0.0494963
841	0.0132401	0.0132401	0.0132401	0.0132401	0.0132401
3281	0.00343006	0.00343006	0.00343006	0.00343006	0.00343006
12961	0.000873305	0.000873305	0.000873305	0.000873305	0.000873305
Union jack pattern, Fig. 9, (d) and (h)					
Dof	$\|\nabla u-G_hu_h\|_{l^2}$	$\|\nabla u-S_hu_h\|_{l^2}$	$\|\nabla u-H_hu_h\|_{l^2}$	$\|\nabla u-A_hu_h\|_{l^2}$	$\|\nabla u-D_hu_h\|_{l^2}$
121	0.0901807	0.0901807	0.0901807	0.0901807	0.0901807
441	0.0255399	0.0255399	0.0255399	0.0255399	0.0255399
1681	0.00714812	0.00714812	0.00714812	0.00714812	0.00714812
6561	0.00203228	0.00203228	0.00203228	0.00203228	0.00203228
25921	0.00060133	0.00060133	0.00060133	0.00060133	0.00060133

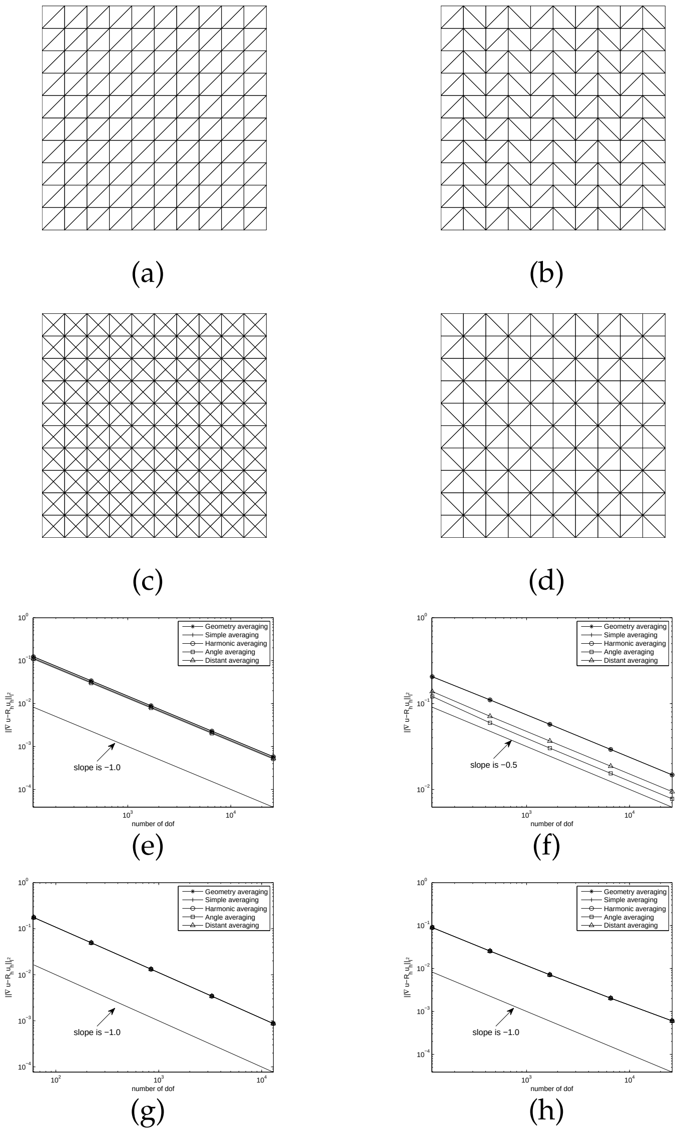


Figure 9: Test case 3 on for uniform mesh patterns; (a), (e) Regular pattern and the results; (b), (f) Chevron pattern and the results; (c), (g) Criss-Cross pattern and the results; (d), (h) Union Jack pattern and the results.

results are interesting. For triangular elements nearly all simple patterns (Regular, Criss-Cross, Union Jack) yield superconvergence for the five weighted averaging recovery methods. The only exception is the Chevron pattern.

6.4 Test case 4

We now consider the five weighted averaging recovery techniques on the nonuniform meshes of various very irregular patterns, see Fig. 10. The results are reported in

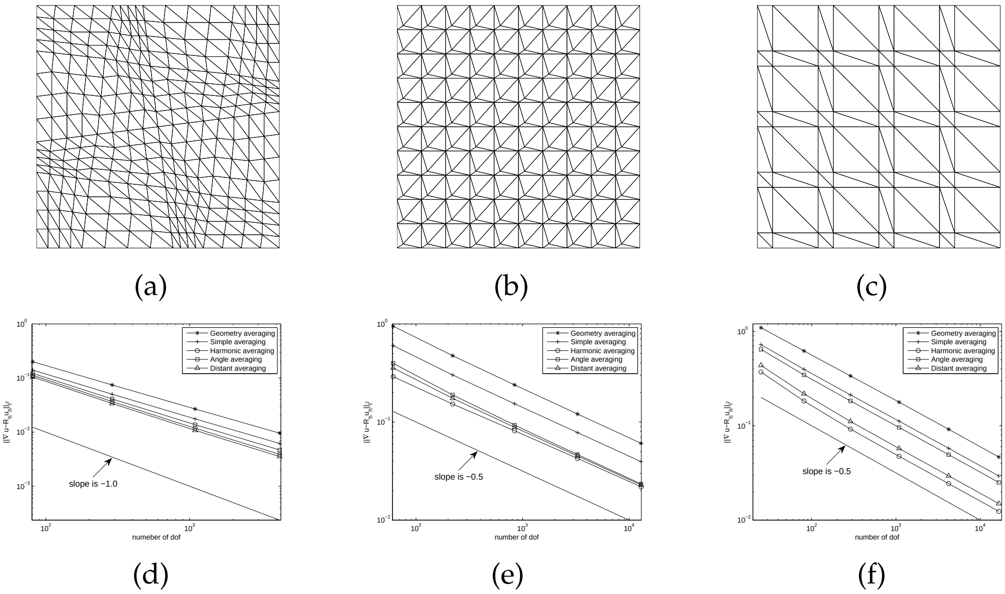


Figure 10: Test case 4 on three nonuniform meshes; (a), (d) Mesh No. 1 and the results; (b), (e) Mesh No. 2 and the results; (c), (f) Mesh No. 3 and the results.

Table 3. Superconvergence is not observed for all the five weighted averaging methods. And among these weighted averaging recovery methods, geometry averaging is worst, and harmonic averaging, angle averaging and distant averaging is better than simple averaging and geometry averaging.

Table 3: Test case 4, results for the finite element solution u_h.

Solution: $u = e^{x^2+y^2}$, nonuniform meshes.					
Mesh No. 1, Fig. 10, (a) and (d)					
Dof	$\|\nabla u - G_h u_h\|_{l^2}$	$\|\nabla u - S_h u_h\|_{l^2}$	$\|\nabla u - H_h u_h\|_{l^2}$	$\|\nabla u - A_h u_h\|_{l^2}$	$\|\nabla u - D_h u_h\|_{l^2}$
81	0.200198	0.14053	0.112383	0.121999	0.104741
289	0.0744506	0.0503046	0.0366306	0.0410282	0.0337959
1089	0.0268373	0.0176104	0.0118249	0.0137613	0.0108148
4225	0.00956679	0.00616213	0.00389798	0.00468494	0.00353932
Mesh No. 2, Fig. 10, (b) and (e)					
Dof	$\|\nabla u - G_h u_h\|_{l^2}$	$\|\nabla u - S_h u_h\|_{l^2}$	$\|\nabla u - H_h u_h\|_{l^2}$	$\|\nabla u - A_h u_h\|_{l^2}$	$\|\nabla u - D_h u_h\|_{l^2}$
61	0.94829	0.600722	0.290334	0.396875	0.356469
221	0.474103	0.301498	0.152531	0.188887	0.174447
841	0.239152	0.153529	0.0813417	0.0931116	0.088407
3281	0.120619	0.0780249	0.0425712	0.0466086	0.045041
12961	0.060662	0.0394224	0.0218591	0.0233907	0.0228219
Mesh No. 3, Fig. 10, (c) and (f)					
Dof	$\|\nabla u - G_h u_h\|_{l^2}$	$\|\nabla u - S_h u_h\|_{l^2}$	$\|\nabla u - H_h u_h\|_{l^2}$	$\|\nabla u - A_h u_h\|_{l^2}$	$\|\nabla u - D_h u_h\|_{l^2}$
25	1.09179	0.723288	0.371101	0.644762	0.436291
81	0.620461	0.396394	0.182665	0.345614	0.218068
289	0.33711	0.212347	0.0921558	0.183157	0.11106
1089	0.177884	0.111761	0.0474857	0.0959118	0.0574116
4225	0.0917676	0.0576794	0.0243624	0.0493832	0.0294794
16641	0.0466617	0.0293485	0.0123773	0.025099	0.0149782

7 Conclusions and future works

We introduce three new weighted averaging methods called harmonic averaging, angle averaging and distant averaging for gradient recovery, respectively. We present analysis and numerical investigation in the performance of these weighted averaging methods under different meshes. It is shown that, harmonic averaging yields a superconvergent gradient recovery for 1D problems and rectangle elements. And under triangular mesh, angle averaging and distant averaging also work well.

The practical usage of recovery technique is not only to improve the quality of the approximation, but also to construct a posteriori error estimators in adaptive computation [1, 2, 21, 24]. Our further investigation will be devoted to analysis of the new weighted averaging recovery methods in application to a posteriori error estimates.

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References

[1] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, New York, Wiley Interscience, 2000.

[2] I. Babuska and T. Strouboulis, The Finite Element Method and Its Reliability, Numerical Mathematics and Scientific Computation, 2001.

[3] R. E. Bank and J. Xu, *Asymptotically exact a posteriori error estimators, part I: grids with superconvergence*, SIAM J. Numer. Anal., 41 (2003), pp. 2294–2312.

[4] J. H. Bramble and A. H. Schatz, *Higher order local accuracy by averaging in the finite element method*, Math. Comput., 31 (1977), pp. 74–111.

[5] C. Carstensen and S. Bartels, *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids: part I: low order conforming, nonconforming, and mixed FEM*, Math. Comput., 71 (2002), pp. 945–969.

[6] C. Carstensen and S. Bartels, *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids: part II: higher order FEM*, Math. Comput., 71 (2002), pp. 971–994.

[7] C. M. Chen and Y. Huang, High Accuracy Theory of Finite Element Methods, Hunan Science Press, Hunan, China (in Chinese), 1995.

[8] B. Heimsund, X. Tai and J. Wang, *Superconvergence for the gradient of the finite element approximations by L^2-projections*, SIAM J. Numer. Anal., 40 (2002), pp. 1538–1560.

[9] E. Hinton, Least Square Analysis Using Finite Elements, M.Sc. Thesis, Civ. Engng. Dept., University College of Swansea, 1968.

[10] E. HINTON AND J. S. CAMPBELL, *Local and global smoothing of discontinuous finite element functions using a least-square method*, Int. J. Numer. Methods. Eng., 8 (1974), pp. 461–480.

[11] I. HLAVACEK, M. KRIZEK AND V. P. PRAHA, *How to recovery the gradient of linear elements on nonuniform triangulations*, App. Mat., 41 (1996), pp. 241–267.

[12] Y. HUANG AND N. YI, *The superconvergent cluster recovery method*, J. Sci. Comput., 44 (2010), pp. 301–322.

[13] B.M. IRONS, *Least square surface fitting by finite elements, and an application to stess smoothing*, Aero. Stree. Memo., ASM 1524, Rolls-Royce, 1967.

[14] Q. LIN AND N. YAN, Construction and Analysis of High Eficient Finite Elements (in Chinese), Hebei University Press, China, 1996.

[15] G. MAISANO, S. MICHELETTI, S. PEROTTO AND C. L. BOTTASSO, *On some new recovery-based a posteriori error estimators*, Comput. Methods. Appl. Mech. Eng., 195 (2006), pp. 4794–4815.

[16] A. NAGA AND Z. ZHANG, *A Posteriori error estimates based on polynomial preserving recovery*, SIAM J. Numer. Anal., 42 (2004), pp. 1780–1800.

[17] J. T. ODEN AND H. J. BRAUCHLI, *On the calculation of consistent stess distributions in finite element applications*, Int. J. Numer. Meth. Eng., 3 (1971), pp. 317–325.

[18] M. TABBARA, T. BLACKER AND T. BELYTSCHKO, *Finite element derivative recovery by moving least square interpolants*, Comput. Methods. Appl. Mech. Eng., 117 (1994), pp. 211–223.

[19] L. B. WAHLBIN, Superconvergence in Galkerkin Finite Element Methods, Springer Verlag, Berlin, 1995.

[20] J. WANG, *A superconvergence analysis for finite element solutions by the least-squares surface fitting on irregular meshes for smooth problems*, J. Math. Study., 33 (2000), pp. 229–243.

[21] J. XU AND Z. M. ZHANG, *Analysis of recovery type a posteriori error estimators for mildly structured grids*, Math. Comput., (2003), pp. 781–801.

[22] N.N. YAN, Superconvergence Analysis and a Posteriori Error Estimation in Finite Element Methods, Science Press, 2008.

[23] Z. ZHANG AND A. NAGA, *A new finite element gradient recovery method: superconvergence property*, SIAM J. Sci. Comput., 26 (2005), pp. 1192–1213.

[24] Z. ZHANG AND J. Z. ZHU, *Superconvergent patch recovery technique and a posteriori error estimator in the finite element method (I)*, Comput. Methods. Appl. Mech. Eng., 123 (1995), pp. 173–187.

[25] J. Z. ZHU AND O. C. ZIENKIEWICZ, *Supercovergence recovery technique and a posteriori error estimates*, Int. J. Numer. Meth. Eng., 30 (1990), pp. 1321–1339.

[26] O. C. ZIENKIEWICZ AND J. Z. ZHU, *The superconvergence patch recovery (SPR) and adaptive finite element refinement*, Comput. Methods. Appl. Mech. Eng., 101 (1992), pp. 207–224.

[27] O. C. ZIENKIEWICZ AND J. Z. ZHU, *The supercovergent patch recovery and a posteriori error estimates*, Int. J. Numer. Methods. Eng., 33 (1992), pp. 1331–1382.