Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 2, pp. 190-204

DOI: 10.4208/aamm.10-m1118
April 2012

Variation of Parameters Method for Solving System of Nonlinear Volterra Integro-Differential Equations

Muhammad Aslam Noor[1,*], Khalida Inayat Noor[1], Asif Waheed[1] and Eisa Al-Said[2]

[1] *Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*
[2] *Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia*

Received 2 February 2011; Accepted (in revised version) 27 May 2011

Available online 26 March 2012

Abstract. It is well known that nonlinear integro-differential equations play vital role in modeling of many physical processes, such as nano-hydrodynamics, drop wise condensation, oceanography, earthquake and wind ripple in desert. Inspired and motivated by these facts, we use the variation of parameters method for solving system of nonlinear Volterra integro-differential equations. The proposed technique is applied without any discretization, perturbation, transformation, restrictive assumptions and is free from Adomian's polynomials. Several examples are given to verify the reliability and efficiency of the proposed technique.

AMS subject classifications: 65N99, 65R20, 65K15

Key words: Variation of parameters method, oceanography, system of nonlinear Volterra integro-differential equations, error estimates.

1 Introduction

It is well-known fact that a wide class of problems in physical and engineering sciences including oceanography, nano-hydrodynamics, drop wise condensation, glass-forming process, and wind ripple in desert can be studied in the general and unified framework of integro-differential equations, see [1–6, 14–33] and the references therein. It has been shown in [7] that fractional order integro-differential equations can be used to model nonlinear oscillations of earthquake. Oceanography is the study of the ocean making use of the various sciences including physics, chemistry, biology, gelogy and mathematics. Physical studies are carried out both by direct observation

*Corresponding author.
Email: noormaslam@hotmail.com (M. Noor), khalidanoor@hotmail.com (K. Noor), waheedasif@hotmai l.com (A. Waheed), eisasaid@ksu.edu.sa (E. Said)

of the properties and movements and also by applying the basic physical principles of mechanics and thermodynamics to determine the motion. The observational approach is known as descriptive or geomathematical oceanography. The dynamical oceanography is used to endeavor to obtain mathematical relations between the forces acting on the ocean water and their consequent motions.

Due to the importance of nonlinear integro-differential equations, several numerical and analytic techniques including modified Adomian's decomposition method [31], Adomian's decomposition method [3], rationalized Haar functions method [14], homotopy peturbation method [1, 4, 5, 19], variational iteration method [15–17, 24–26] and modified variation of parameters method [20] have been developed. Wazwaz [32] used Modified Adomian's decomposition method to solve some integro-differential equations related to Blasius problems. Sayed et al. [3] applied decomposition method to solve linear Voltera Fredholm integro-differential equations. Maleknejad et al. [14] solved system of linear integro-differential equations by using rationalized Haar functions method and Biazar et al. [1] applied homotoppy perturbation method to solve nonlinear system of integro-differential equations. Ghasemi et al. [5] and Yusufoglu [34] used homotopy perturbation method for solving Volterra integro-differential equations. Wang et. al. [31] and Nadjafi et al. [18] applied variational iteration method to solve system of nonlinear integro-differential equations. Mohyud-Din et al. [16] have solved nonlinear system of integro-differential equations by modified variation of parameters method in which he coupled both homotopy perturbation method and variation of parameters method. Most of these methods have their inbuilt deficiencies like calculation of Adomian's polynomials, use of small parameters, identification of Lagrange multiplier, divergent results and huge computational work. These facts motivated us to consider variation of parameters method [8–10, 15–17, 24–26] for solving system of nonlinear integro-differential equations. This technique is a very useful tool in analytic studies and helps to improve our understanding of what dynamical effects may be important. The use of multiplier in variation of parameters method increase the rate of convergence by reducing the number of iterations, reduce the successive applications of integral operator and make the solution procedure simple while still maintaining a very high level of accuracy. The multiplier used in variation of parameters method is obtained by Wronskian technique and is totally different from Lagrange multiplier of variational iteration method.

Moreover, variation of parameters method removes the higher order derivative term from its iterative scheme which is clear advantage over the variational iteration method as the term may cause of repeated computation and calculations of unneeded terms, which consumes both the time and effort, in most of the cases. Thus variation of parameters method has reduced lot of computational work involved due to this term as compared to some other existing techniques using this term which is clear advantage of proposed technique over them. Hence, variation of parameters method provides a wider and better applicability as compare to other classical techniques. Ma et al. [8–10] presented variation of parameters method to solve some nonhomogenous partial differential equations. Ramos [28] used variation of parameters method to find frequency of some non linear oscillators.

In this paper, we have applied variation of parameters method to solve systems of second-order nonlinear Volterra integro-differential equations. In Section 2, we discuss the derivation of the variation of parameters method to convey the idea. Some examples are considered in Section 3 to illustrate the implementation and efficiency of the variation of parameters method. Results are quite encouraging and may stimulate further research in the applications of this method.

2 Variation of parameters method

To convey the basic idea of the variation of parameters method for differential equations, we consider the general differential equation of the form

$$Lu(x)+Ru(x)+Nu(x)=g(x), \tag{2.1}$$

where L is the linear operator, R is a linear partial operator of order less than L, N is a nonlinear operator and g is a source term. Using variation of parameters method [15–17,24–26], one can obtain the general solution of Eq. (2.1) in the following form:

$$u(x)=\sum_{i=0}^{n-1}\frac{B_i x^i}{i!}+\int_0^x \lambda(x,s)\big(-Nu(s)-Ru(s)+g(s)\big)ds, \tag{2.2}$$

where n is a order of given differential equation and $B_i{}'s$ are unknowns which can be further determined by initial/boundary conditions. Here $\lambda(x,s)$ is multiplier which can be obtained with the help of Wronskian technique. This multiplier removes the successive applications of integral in iterative scheme and it depends upon the order of equation. Noor et al. [16] have obtained the multiplier $\lambda(x,s)$ in the form:

$$\lambda(x,s)=\sum_{i=1}^{n}\frac{s^{i-1}x^{n-i}(-1)^{i-1}}{(i-1)!(n-i)!}. \tag{2.3}$$

For different choices of n, one can obtain the following values of λ

$$\begin{aligned}
n&=1, &\quad \lambda(x,s)&=1,\\
n&=2, &\quad \lambda(x,s)&=x-s,\\
n&=3, &\quad \lambda(x,s)&=\frac{x^2}{2!}-sx+\frac{s^2}{2!},\\
n&=4, &\quad \lambda(x,s)&=\frac{x^3}{3!}-\frac{sx^2}{2!}+\frac{s^2x}{2!}-\frac{s^3}{3!},\cdots.
\end{aligned}$$

Hence, we have the following iterative scheme from (2.2)

$$u_{k+1}(x)=u_k(x)+\int_0^x \lambda(x,s)\big(-Nu_k(s)-Ru_k(s)+g(s)\big)ds, \quad k=0,1,2,\cdots. \tag{2.4}$$

It is observed that the fix value of initial guess in each iteration provides the better approximation, that is, $u_k(x) = u_0(x)$, for $k = 1, 2, \cdots$. However, we can modify the initial guess by dividing $u_0(x)$ in two parts and using one of them as initial guess. It is more convenient way in case of more than two terms in $u_0(x)$.

Now for solving system of nonlinear system of integro-differential equations, we have following iterative scheme

$$y_{i,k+1}(x) = y_{k,i}(x) + \int_a^x \lambda(x,s)\Big(H_j\big(s, y_{i,k}(s), \cdots, y_{i,k}^{(m)}(s)\big) + \int_a^s K_j\big(s, t, y_{i,k}(t), \cdots, y_{i,k}^{(m)}(t)\big)dt\Big)ds, \tag{2.5}$$

where $i, j = 1, 2, \cdots, n$ and $k = 0, 1, 2, \cdots$.

3 Numerical applications

In this section, we have applied variation of parameter method to solve system of second-order nonlinear Volterra integro-differential equations. Such type integro-differential equations arise in many physical processes, such as oceanography, nano-hydrodynamics, drop wise condensation, earthquake, glass-forming process, and wind ripple in desert, see [1–7, 13, 16–34] and references there in. For the comparison purpose, we consider the same examples as in [1, 16, 32].

Example 3.1. (see [1, 16]). Consider the system of second-order nonlinear integro-differential equations as follows:

$$\begin{aligned} u''(x) &= 1 - \frac{1}{3}x^3 - \frac{1}{2}v'^2(x) + \frac{1}{2}\int_0^x \big(u^2(t) + v^2(t)\big)dt, \\ v''(x) &= -1 + x^2 - xu(x) + \frac{1}{4}\int_0^x \big(u^2(t) - v^2(t)\big)dt, \end{aligned}$$

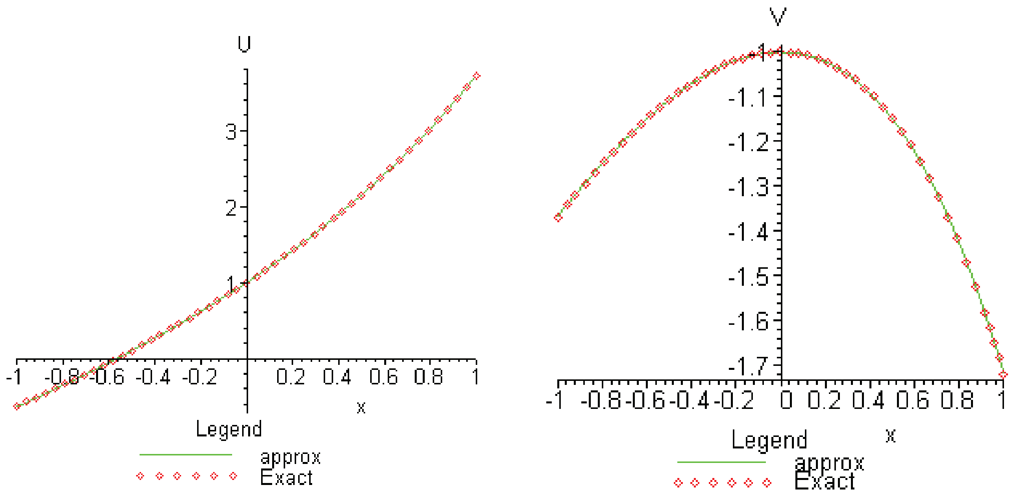
with initial conditions $u(0) = 1$, $u'(0) = 2$, $v(0) = -1$, $v'(0) = 0$.

The exact solutions for this problem are $u(x) = x + e^x$, $v(x) = x - e^x$. Applying the variation of parameters method, we have

$$\begin{aligned} u_{k+1}(x) &= A_1 + A_2 x + \int_0^x \lambda(x,s)\Big(1 - \frac{1}{3}s^3 - \frac{1}{2}{v'_k}^2(s) + \frac{1}{2}\int_0^s ({u_k}^2(t) + {v_k}^2(t))dt\Big)ds, \\ v_{k+1}(x) &= B_1 + B_2 x + \int_0^x \lambda(x,s)\Big(1 - \frac{1}{3}s^3 - \frac{1}{2}{v'_k}^2(s) + \frac{1}{2}\int_0^s ({u_k}^2(t) + {v_k}^2(t))dt\Big)ds. \end{aligned}$$

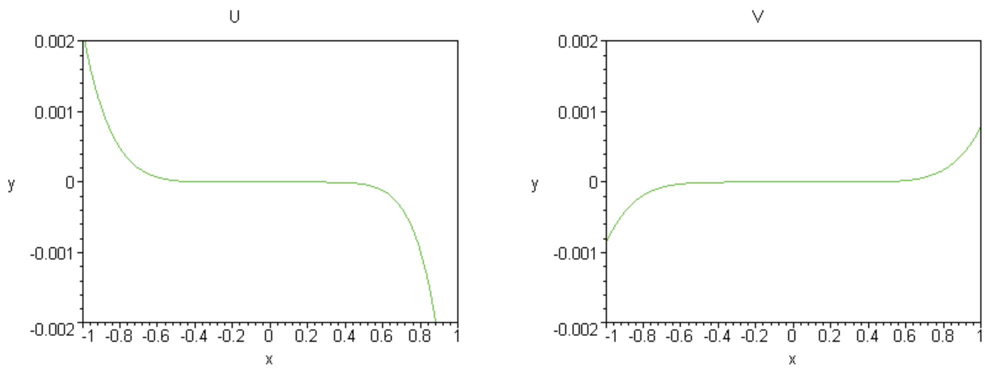
Using $\lambda(x,s) = x - s$, since the governing equation are of 2^{nd} order

$$\begin{aligned} u_{k+1}(x) &= A_1 + A_2 x + \int_0^x (x-s)\Big(1 - \frac{1}{3}s^3 - \frac{1}{2}{v'_k}^2(s) + \frac{1}{2}\int_0^s ({u_k}^2(t) + {v_k}^2(t))dt\Big)ds, \\ v_{k+1}(x) &= B_1 + B_2 x + \int_0^x (x-s)\Big(1 - \frac{1}{3}s^3 - \frac{1}{2}{v'_k}^2(s) + \frac{1}{2}\int_0^s ({u_k}^2(t) + {v_k}^2(t))dt\Big)ds. \end{aligned}$$



(a) (b)

Figure 1: Example 3.1: Graphical comparison between exact solution and approximate solution.



(a) (b)

Figure 2: Example 3.1: Error estimates.

Using the initial conditions, we have

$$A_1 = 1, \qquad A_2 = 2, \qquad B_1 = -1, \qquad B_2 = 0.$$

Hence, we have following iterative scheme

$$\begin{aligned} u_{k+1}(x) &= u_0(x) + \int_0^x (x-s)\Big(1 - \frac{1}{3}s^3 - \frac{1}{2}{v'_k}^2(s) + \frac{1}{2}\int_0^s ({u_k}^2(t) + {v_k}^2(t))dt\Big)ds, \\ v_{k+1}(x) &= v_0(x) + \int_0^x (x-s)\Big(1 - \frac{1}{3}s^3 - \frac{1}{2}{v'_k}^2(s) + \frac{1}{2}\int_0^s ({u_k}^2(t) + {v_k}^2(t))dt\Big)ds. \end{aligned}$$

Consequently, we have the following approximations

$$\begin{aligned} u_0(x) &= 1 + 2x, \\ v_0(x) &= -1, \end{aligned}$$

Table 1: Example 3.1: Error estimates.

x	Errors U	Errors V
-1.0	2.16E-03	8.74E-04
-0.8	4.80E-04	1.85E-04
-0.6	6.87E-05	2.49E-05
-0.4	4.35E-06	1.46E-06
-0.2	3.71E-08	1.15E-08
0.0	0.00000	0.00000
0.2	4.50E-08	1.10E-08
0.4	6.27E-06	1.42E-06
0.6	1.17E-04	2.38E-05
0.8	9.69E-04	1.73E-04
1.0	5.06E-03	7.89E-04

$$
\begin{aligned}
u_1(x) &= 1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5,\\
v_1(x) &= -1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{60}x^5,\\
u_2(x) &= 1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{17}{5040}x^7+\frac{1}{672}x^8\\
&\quad+\frac{53}{120960}x^9+\frac{1}{103680}x^{10}+\frac{1}{228096}x^{11}+\frac{1}{900800}x^{12}+\frac{1}{6177600}x^{13},\\
v_2(x) &= -1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5-\frac{1}{720}x^6-\frac{11}{10080}x^7+\frac{13}{241920}x^9\\
&\quad+\frac{17}{1036800}x^{10}+\frac{47}{11404800}x^{11}+\frac{1}{1267200}x^{12}.
\end{aligned}
$$

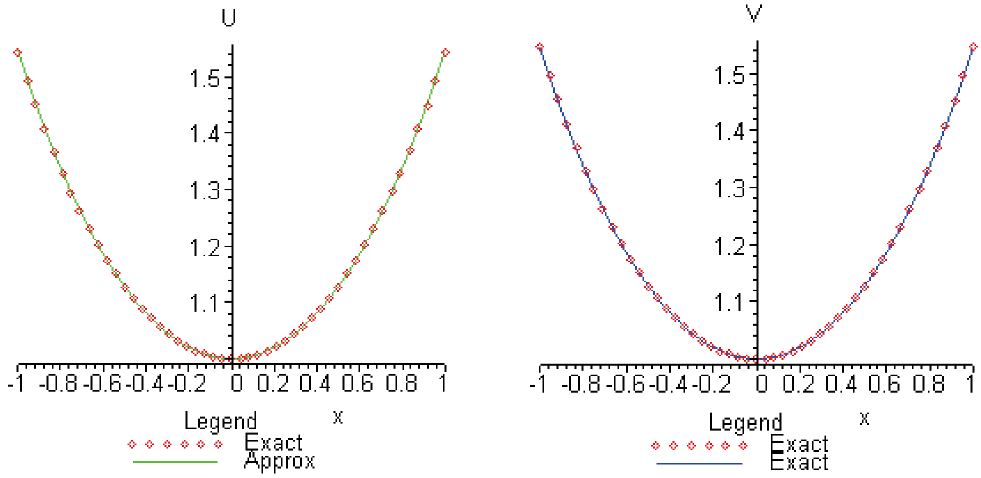
The series solution is given by

$$
\begin{aligned}
u(x) &= 1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{60}x^5-\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{17}{5040}x^7+\frac{1}{672}x^8\\
&\quad+\frac{53}{120960}x^9-\frac{1}{103680}x^{10}+\frac{17}{228096}x^{11}+\frac{1}{1900800}x^{12}+\frac{1}{6177600}x^{13}+\cdots,\\
v(x) &= -1-\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{60}x^5\frac{1}{120}x^5-\frac{1}{720}x^6-\frac{11}{10080}x^7+\frac{13}{241920}x^9\\
&\quad+\frac{17}{1036800}x^{10}+\frac{47}{11404800}x^{11}+\frac{1}{1267200}x^{12}+\cdots.
\end{aligned}
$$

Example 3.2. (see [1,16]). Consider the system of nonlinear integro-differential equations as follows:

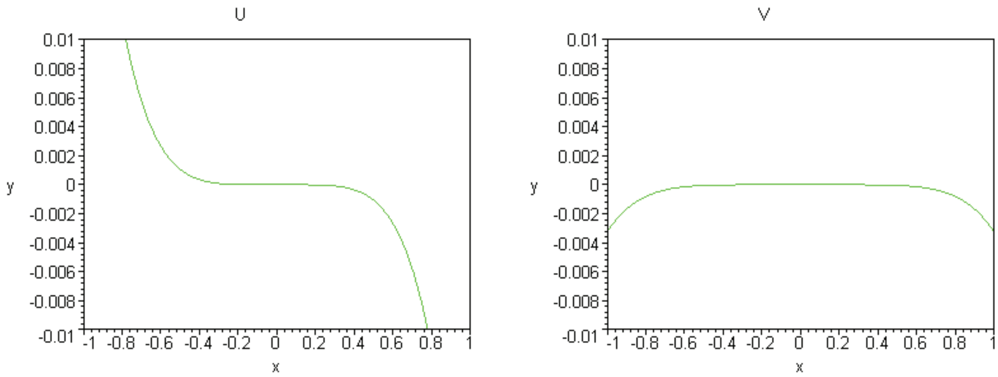
$$
\begin{aligned}
u'(x) &= 1-\frac{1}{2}v'^2(x)+\frac{1}{2}\int_0^x \big((x-t)v(t)+u(t)v(t)\big)dt,\\
v'(x) &= 2x+\int_0^x \big((x-t)u(t)-v^2(t)+u^2(t)\big)dt,
\end{aligned}
$$

with initial conditions $u(0)=0$, $v(0)=1$.



(a) (b)

Figure 3: Example 3.2: Graphical comparison between exact solution and approximate solution.



(a) (b)

Figure 4: Example 3.2: Error estimates.

The exact solutions for this problem are $u(x) = \sinh x$, $v(x) = \cosh x$. Applying the variation of parameters method, we have

$$u_{k+1}(x) = A_1 + \int_0^x \lambda(x,s)\Big(1 - \frac{1}{2}v'^2(s) + \frac{1}{2}\int_0^s ((s-t)v(t) + u(t)v(t))dt\Big)ds,$$
$$v_{k+1}(x) = B_1 + \int_0^x \lambda(x,s)\Big(2s + \int_0^x ((s-t)u(t) - v^2(t) + u^2(t))dt\Big)ds.$$

Using $\lambda(x,s) = 1$, since the governing equation are of 1^{st} order

$$u_{k+1}(x) = A_1 + \int_0^x \Big(1 - \frac{1}{2}v'^2(s) + \frac{1}{2}\int_0^s ((s-t)v(t) + u(t)v(t))dt\Big)ds,$$
$$v_{k+1}(x) = B_1 + \int_0^x \Big(2s + \int_0^x ((s-t)u(t) - v^2(t) + u^2(t))dt\Big)ds.$$

Table 2: Example 3.2: Error estimates.

x	Errors U	Errors V
-1.0	3.51E-02	3.24E-02
-0.8	1.12E-02	8.07E-02
-0.6	2.64E-03	1.37E-03
-0.4	3.44E-04	1.16E-04
-0.2	1.06E-05	1.79E-05
0.0	0.00000	0.00000
0.2	1.06E-05	1.79E-05
0.4	3.44E-04	1.16E-04
0.6	2.64E-03	1.37E-03
0.8	3.12E-02	8.07E-02
1.0	3.51E-02	3.24E-02

Using the initial conditions, we have

$$A_1 = 0, \qquad B_1 = 1.$$

Hence, we have following iterative scheme

$$\begin{aligned} u_{k+1}(x) &= u_0(x) + \int_0^x \Big(1 - \frac{1}{2}v'^2(s) + \frac{1}{2}\int_0^s ((s-t)v(t) + u(t)v(t))dt\Big)ds, \\ v_{k+1}(x) &= u_0(x) + \int_0^x \Big(2s + \int_0^x ((s-t)u(t) - v^2(t) + u^2(t))dt\Big)ds. \end{aligned}$$

Consequently, we have the following approximations

$$\begin{aligned} u_0(x) &= 0, \\ v_0(x) &= 1, \\ u_1(x) &= x + \frac{1}{6}x^3, \\ v_1(x) &= 1 + \frac{1}{2}x^2, \\ u_2(x) &= x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{504}x^7, \\ v_2(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{240}x^6 + \frac{1}{2016}x^8. \end{aligned}$$

The series solution is given by

$$\begin{aligned} u(x) &= x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{504}x^7 + \cdots, \\ v(x) &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{240}x^6 + \frac{1}{2016}x^8 + \cdots. \end{aligned}$$

Example 3.3. (see [1,16]). Consider the following another nonlinear system of three integro-differential equations:

$$
\begin{aligned}
u''(x) &= x + 2x^3 + 2v'^2(x) - \int_0^x \big(v'^2(t) + u(t)w''(t)\big)dt,\\
v''(x) &= 3x^2 - xu(x) + \int_0^x \big(txv'(t)u''(t) - w'(t)\big)dt,\\
w''(x) &= 2 - \frac{4}{3}x^3 + u''^2(x) - 2u^2(x) + \int_0^x \big(x^2 v(t) + u'^2(t) + t^3 w''(t)\big)dt,
\end{aligned}
$$

with initial conditions

$$
u(0) = 1, \quad u'(0) = 0, \quad v(0) = 0, \quad v'(0) = 1, \quad w(0) = 0, \quad w'(0) = 0.
$$

The exact solutions for this problem are $u(x) = x^2$, $v(x) = x$, $w(x) = 3x^2$. Applying the variation of parameters method, we have

$$
\begin{aligned}
u_{k+1}(x) &= A_1 + A_2 x + \int_0^x \lambda(x,s)\Big(s + 2s^3 + 2{v'_k}^2(s) - \int_0^s ({v'_k}^2(t) + u_k(t){w''}_k(t))dt\Big)ds,\\
v_{k+1}(x) &= B_1 + B_2 x + \int_0^x \lambda(x,s)\Big(3s^2 - su_k(s) + \int_0^s (tsv'_k(t){u''}_k(t) - w'_k(t))dt\Big)ds,\\
w''(x) &= C_1 + C_2 x + \int_0^x \lambda(x,s)\Big(2 - \frac{4}{3}s^3 + {u''_k}^2(s) - 2{u_k}^2(s)\\
&\quad + \int_0^s (s^2 v(t) + u'^2(t) + t^3 w''(t))dt\Big)ds.
\end{aligned}
$$

Using $\lambda(x,s) = x - s$, since the governing equations are of 2^{nd} order

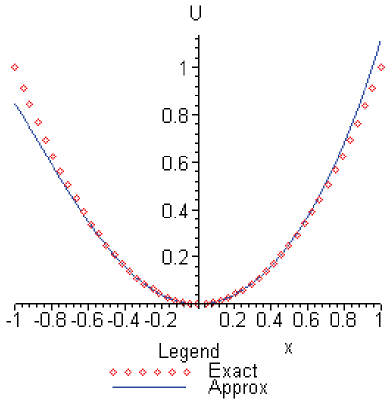
$$
\begin{aligned}
u_{k+1}(x) &= A_1 + A_2 x + \int_0^x (x-s)\Big(s + 2s^3 + 2{v'_k}^2(s) - \int_0^s ({v'_k}^2(t) + u_k(t){w''}_k(t))dt\Big)ds,\\
v_{k+1}(x) &= B_1 + B_2 x + \int_0^x (x-s)\Big(3s^2 - su_k(s) + \int_0^s (tsv'_k(t){u''}_k(t) - w'_k(t))dt\Big)ds,\\
w''(x) &= C_1 + C_2 x + \int_0^x (x-s)\Big(2 - \frac{4}{3}s^3 + {u''_k}^2(s) - 2{u_k}^2(s) + \int_0^s (s^2 v(t)\\
&\quad + u'^2(t) + t^3 w''(t))dt\Big)ds.
\end{aligned}
$$

Using the initial conditions, we have

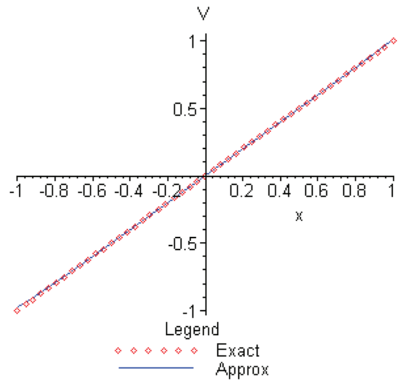
$$
A_1 = 1, \qquad A_2 = 0, \qquad B_1 = 0, \qquad B_2 = 1, \qquad C_1 = 0, \qquad C_2 = 0.
$$

And hence, we have following iterative scheme

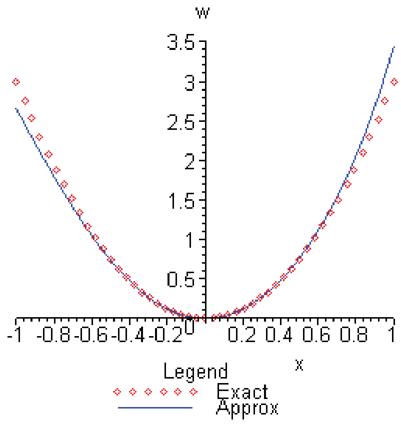
$$
\begin{aligned}
u_{k+1}(x) &= u_0(x) + \int_0^x (x-s)\Big(s + 2s^3 + 2{v'_k}^2(s) - \int_0^s ({v'_k}^2(t) + u_k(t){w''}_k(t))dt\Big)ds,\\
v_{k+1}(x) &= v_0(x) + \int_0^x (x-s)\Big(3s^2 - su_k(s) + \int_0^s (tsv'_k(t){u''}_k(t) - w'_k(t))dt\Big)ds,\\
w_{k+1}(x) &= w_0(x) + \int_0^x (x-s)\Big(2 - \frac{4}{3}s^3 + {u''_k}^2(s) - 2{u_k}^2(s)\\
&\quad + \int_0^s (s^2 v(t) + u'^2(t) + t^3 w''(t))dt\Big)ds.
\end{aligned}
$$



(a)



(b)

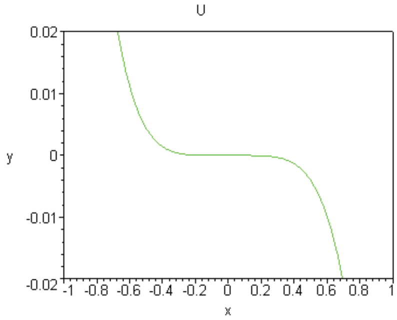


(c)

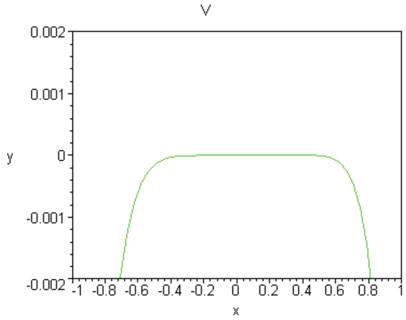
Figure 5: Example 3.3: Graphical comparison between exact solution and approximate solution.

Consequently, we have the following approximations

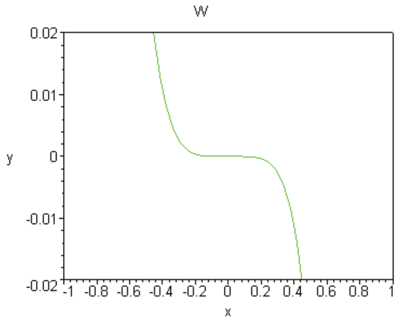
$$
\begin{aligned}
u_0(x) &= 0,\\
v_0(x) &= x,\\
w_0(x) &= 0,\\
u_1(x) &= x^2 + \frac{1}{10}x^5,\\
v_1(x) &= x + \frac{1}{4}x^4,\\
w_1(x) &= x^2 - \frac{1}{15}x^5 + \frac{1}{60}x^6,\\
u_2(x) &= x^2 + \frac{4}{15}x^5 - \frac{1}{60}x^6 + \frac{197}{5040}x^8 - \frac{1}{336}x^9 + \frac{1}{7425}x^{11} + \frac{1}{26400}x^{12},\\
v_2(x) &= x + \frac{1}{6}x^4 + \frac{1}{630}x^7 + \frac{41}{3360}x^8 + \frac{1}{440}x^{11},\\
w_2(x) &= 3x^2 + \frac{2}{5}x^5 - \frac{1}{30}x^6 + \frac{13}{168}x^8 - \frac{227}{30240}x^9 + \frac{1}{1440}x^{10} + \frac{1}{3960}x^{11} - \frac{1}{6600}x^{12}, \cdots .
\end{aligned}
$$



(a)



(b)



(c)

Figure 6: Example 3.3: Error estimates.

Table 3: Example 3.3: Error estimates.

	Errors		
x	U	V	W
-1.	2.33E-02	3.48E-02	3.48E-01
-0.8	4.79E-03	1.25E-03	1.25E-01
-0.6	5.77E-04	3.12E-04	3.12E-02
-0.4	2.83E-05	4.17E-05	4.17E-03
-0.2	1.72E-07	1.29E-07	1.29E-04
0.0	0.00000	0.00000	0.00000
0.2	7.20E-08	1.26E-08	1.26E-04
0.4	2.80E-06	4.00E-06	4.00E-03
0.6	7.15E-05	3.07E-05	3.07E-02
0.8	1.64E-03	1.34E-03	1.34E-01
1.0	1.53E-02	4.37E-02	4.37E-01

The series solution is given by

$$u(x) = x^2 + \frac{4}{15}x^5 - \frac{1}{60}x^6 + \frac{197}{5040}x^8 - \frac{1}{336}x^9 + \frac{1}{7425}x^{11} + \frac{1}{26400}x^{12} + \cdots,$$

$$v(x) = x + \frac{1}{6}x^4 + \frac{1}{630}x^7 + \frac{41}{3360}x^8 + \frac{1}{440}x^{11} + \cdots,$$

$$w(x) = 3x^2 + \frac{2}{5}x^5 - \frac{1}{30}x^6 + \frac{13}{168}x^8 - \frac{227}{30240}x^9 + \frac{1}{1440}x^{10} + \frac{1}{3960}x^{11} - \frac{1}{6600}x^{12} + \cdots.$$

Example 3.4. (see [16,32]). Consider the two dimensional nonlinear inhomogeneous initial boundary value problem for the integro differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2}\int_0^x y(t)y''(t)dt, \quad -\infty < x < 0,$$

with initial conditions $y(0) = 0$, $y'(0) = 1$, $\lim_{x\to\infty} y'(x) = 0$, where constant α is positive and defined by $y''(0) = \alpha$ and $\alpha > 0$.

Applying the variation of parameters method, we have

$$y_{k+1}(x) = A_1 + A_2 x + \int_0^x \lambda(x,s)\Big(\alpha - \int_0^s (y_k(t){y_k}''(t))dt\Big)ds.$$

Using $\lambda(x,s) = x - s$, since the governing equations is of second-order

$$y_{k+1}(x) = A_1 + A_2 x + \int_0^x (x-s)\Big(\alpha - \int_0^s (y_k(t){y_k}''(t))dt\Big)ds.$$

Using the initial conditions, we have

$$A_1 = 0, \qquad A_2 = 1.$$

Hence, we have following iterative scheme

$$y_{k+1}(x) = x + \int_0^x (x-s)\Big(\alpha - \int_0^s (y_k(t){y_k}''(t))dt\Big)ds.$$

Consequently, we have the following approximations

$$\begin{aligned}
y_0(x) &= x,\\
y_1(x) &= x + \frac{1}{2}\alpha x^2,\\
y_2(x) &= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5,\\
y_3(x) &= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 + \frac{11}{161280}\alpha^3 x^8\\
&\quad - \frac{1}{193536}\alpha^2 x^9 - \frac{1}{5702400}\alpha^3 x^{10} - \frac{5}{4257792}\alpha^4 x^{11}.
\end{aligned}$$

The series solution is given as

$$\begin{aligned}
y(x) = x &+ \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 + \frac{11}{161280}\alpha^3 x^8\\
&- \frac{1}{193536}\alpha^2 x^9 - \frac{1}{5702400}\alpha^3 x^{10} - \frac{5}{4257792}\alpha^4 x^{11} + \cdots,
\end{aligned}$$

and consequently

$$\begin{aligned}
y'(x) = 1 + \alpha x &- \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 + \frac{11}{2880}\alpha^2 x^6 + \frac{11}{20160}\alpha^3 x^7 - \frac{11}{21504}\alpha^2 x^8\\
&- \frac{1}{51840}\alpha^3 x^9 - \frac{1}{518400}\alpha^4 x^{10} + 12\Big(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3\Big)x^{11} + \cdots.
\end{aligned}$$

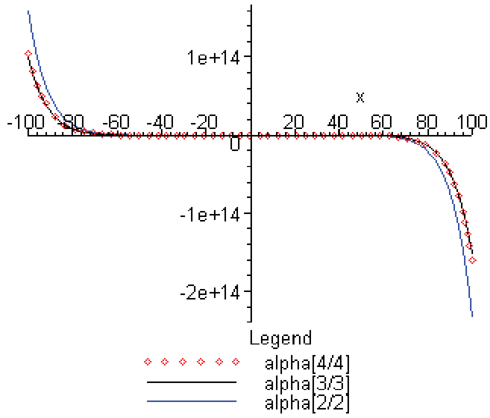


Figure 7: Example 3.4: Graphical representation of solution for different α's.

Table 4: Example 3.4: Padé approximants and numerical value of α.

Padé approximant	α
[2/2]	0.5773502693
[3/3]	0.5163977793
[4/4]	0.5227030798

The diagonal Padé approximants can be applied to determine a numerical value for the constant α by using the given condition.

Remark 3.1. We would also like to mention that Ma et al. see [11–13] have used the multiple exp-function method, linear superposition principle and transformed rational function technique for solving the Hirota bilinear equations for constructing a general class of exact solutions including N-soliton solutions and other aspects of the nonlinear equations. The multiple exp-function method is a kind of generalization of Hirota's bilinear method. It is an interesting and open problem to compare the modified variation of parameters method with the techniques of Ma et al. see [11–13] for solving the system of nonlinear boundary value problems and system of nonlinear Volterra integro-differential equations.

4 Conclusions

In this paper, we have applied the variation of parameters method for solving the nonlinear system of integro-differential equations which arise in modeling of various physical processes including nano hydrodynamics, earthquake, wind ripple in desert and oceanography. The proposed technique is employed without linearization, perturbation, discretization and restrictive assumptions. We would like to emphasize that the suggested method is free from round off errors, identification of Lagrange multiplier and use of Adomian's polynomials. It may be concluded that the variation of parameters method is very powerful and efficient technique in finding the analytical solutions for a wide class of system of integro-differential equations and can be considered as a best alternative of existing iterative techniques for solving such problems.

Acknowledgments

The authors would like to thank the referee and the editor for the very constructive suggestions and comments. This research is supported by the Visiting Professor Program of King Saud University, Riyadh, Saudi Arabia and Research grant No. KSU.-VPP.108.

References

[1] J. BIAZAR, H. GHAZVINI AND M. ESLAMI, *He's homotopy perturbation method for system of integro-differential equations*, Chaos. Solit. Fract., 39 (2009), pp. 1253–1258.

[2] T. L. BO, L. XIE, X AND J. ZHENG, *Numerical approach to wind ripple in dessert*, Int. J. Nonl. Sci. Num. Sim., 8(2) (2007), pp. 223–228.

[3] S. M. EL-SAYED, D. KAYA AND S. ZAREA, *The decomposition method applied to solve high order linear Voltera Fredholm integro-differential equations*, Int. J. Nonl. Sci. Num. Sim., 5(2) (2004), pp. 105–112.

[4] M. EL-SHAHED, *Application of He's homotopy perturbation method to Voltera's integro-differential equation*, Int. J. Nonl. Sci. Num. Sim., 6(2) (2005), pp. 163–168.

[5] M. GHASEMI, M. T. KAJANI AND E. BABOLIAN, *Application of He's homotopy perturbation method to nonlinear integro differential equations*, Appl. Math. Comput., 188 (2007), pp. 538–548.

[6] S. GOGHARY, H. S. JAVADI AND E. BABOLIAN, *Restarted Adomian method for system of nonlinear Volterra integral equations*, Appl. Math. Comput., 161 (2005), pp. 745–751.

[7] J. H. HE, *Nonlinear oscillation with fractional derivative and its applications*, in: International Conference on Vibrating Engineering' 98, Dalian, China, 1998, pp. 288–291.

[8] W. X. MA AND Y. YOU, *Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions*, Trans. Am. Math. Soc., 357 (2004), pp. 1753–1778.

[9] W. X. MA AND Y. YOU, *Rational solutions of the Toda lattice equation in casoratian form*, Chaos. Solit. Fract., 22 (2004), pp. 395–406.

[10] W. X. MA, C. X. LI AND J. S. HE, *A second Wronskian formulation of the Boussinesq equation*, Nonl. Anal., 70(12) (2008), pp. 4245–4258.

[11] W. X. MA AND E. FAN, *Linear superposition principle applying to Hirota bilinear equations*, Comput. Math. Appl., 61 (2011), pp. 950–959.

[12] W. X. MA, T. HUANG AND Y. ZHANG, *A multiple exp-function method for nonlinear differential equations and its applications*, Phys. Scr., 82 (2010), 065003.

[13] W. X. MA, *A transformed rational function method and exact solutions to the 3+1 dimensional Jimbo-Miwa equation*, Chaos. Solit. Fract., 42 (2009), pp. 1356–1363.

[14] K. MALEKNEJAD, F. MIRZAEE AND S. ABBASBANDY, *Solving linear integro-differential equations system by using rationalized Haar functions method*, Appl. Math. Comput., 155 (2004), pp. 17–328.

[15] S. T. MOHYUD-DIN, M. A. NOOR, K. I. NOOR AND A. WAHEED, *Modified variation of parameters method for solving nonlinear boundary value problems*, Int. J. Mod. Phys. B., (2011), in press.

[16] S. T. MOHYUD-DIN, M. A. NOOR AND K. I. NOOR, *Modified variation of parameters method for second-order integro-differential equations and coupled systems*, World. Appl. Sci. J., 6(8) (2009), pp. 1139–1146.

[17] S. T. MOHYUD-DIN, M. A. NOOR AND A. WAHEED, *Variation of parameters method for solving sixth-order boundary value problems*, Commun. Korean. Math. Soc., 24(4) (2009), pp. 605–615.

[18] J. S. NADJAFI AND M. TAMAMGAR, *The variational iteration method, a highly promising method for system of integro-differential equations*, Comput. Math. Appl., 56 (2008), pp. 346–351.

[19] M. A. NOOR, *Iterative methods for nonlinear equations using homotopy perturbation method*, Appl. Math. Inf. Sci., 4(2) (2010), pp. 2227–2235.

[20] M. A. NOOR, *Variational inequalities in physical oceanography*, in: Ocean Wave Engineering, (edit. M. Rahman), Computational Mechanics Publications, Southampton, UK, 1994, pp. 201–226.

[21] M. A. NOOR, *Some developments in general variaitonal inequalities*, Appl. Math. Comput., 152 (2004), pp. 199–277.

[22] M. A. NOOR, *Extended general variational inequalities*, Appl. Math. Lett., 22 (2009), pp. 182–185.

[23] M. A. NOOR, K. I. NOOR AND TH. M. RASSIAS, *Some aspects of variational inequalities*, J. Comput. Appl. Math., 47 (1993), pp. 285–312.

[24] M. A. NOOR, S. T. MOHYUD-DIN AND A. WAHEED, *Variation of parameters method for solving fifth-order boundary value problems*, Appl. Math. Inf. Sci., 2 (2008), pp. 135–141.

[25] M. A. NOOR, K. I. NOOR, A. WAHEED AND E. AL-SAID, *Modified variation of parameters method for solving a system of second-order nonlinear boundary value problems*, Int. J. Phys. Sci., 5(16) (2010), pp. 2426–2431.

[26] M. A. NOOR, K. I. NOOR, A. WAHEED AND E. AL-SAID, *On computation methods for solving systems of fourth-order nonlinear boundary value problems*, Int. J. Phys. Sci,. 6(1) (2011), pp. 128–135.

[27] M. A. NOOR, K. I. NOOR, A. WAHEED AND E. AL-SAID, *Variation of parameters method for solving a class of eight-order boundary value problems*, Int. J. Comput. Methods., 8 (2011), in press.

[28] J. I. RAMOS, *On the variational iteration method and other iterative techniques for nonlinear differential equations*, Appl. Math. Comput., 199 (2008), pp. 39–69.

[29] F. Z. SUN, M. GAO AND S. H. LEI, *The fractal dimension of fractal model of dropwise condensation and its experimental study*, Int. J. Nonl. Sci. Num. Sim., 8(2) (2007), pp. 211–222.

[30] H. WANG, H. M. FU AND H. F. ZHANG, *A practical thermodynamic method to calculate the best glass-forming composition for bulk metallic glasses*, Int. J. Nonl. Sci. Num. Sim., 8(2) (2007), pp. 171–178.

[31] S. Q. WANG AND J. H. HE, *Variational iteration method for solving integro-differential equations*, Phys. Lett. A., 367 (2007), pp. 188–191.

[32] A. M. WAZWAZ, *A reliable algorithm for solving boundary value problems for higher-order integro-differential equations*, Appl. Math. Comput., 118 (2001), pp. 327–342.

[33] L. XU, J. H. HE AND Y. LIU, *Electrospun nanoporous spheres with Chinese drug*, Int. J. Nonl. Sci. Num. Sim., 8(2) (2007), pp. 191–202.

[34] E. YUSUFOGLU, *An efficient algorithm for solving integro-differential equations system*, Appl. Math. Comput., 192(2) (2007), pp. 51–55.