Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 2, pp. 223-237

DOI: 10.4208/aamm.10-m1071
April 2012

The Pullback Asymptotic Behavior of the Solutions for 2D Nonautonomous *G*-Navier-Stokes Equations

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Received 26 December 2010; Accepted (in revised version) 14 December 2011

Available online 26 March 2012

Abstract. The pullback asymptotic behavior of the solutions for 2D Nonautonomous G-Navier-Stokes equations is studied, and the existence of its L^2-pullback attractors on some bounded domains with Dirichlet boundary conditions is investigated by using the measure of noncompactness. Then the estimation of the fractal dimensions for the 2D G-Navier-Stokes equations is given.

AMS subject classifications: 76D05, 37L30, 37B55

Key words: Pullback attractor, G-Navier-Stokes equation, fractal dimension, the measure of noncompactness, bounded domains.

1 Introduction

The Navier-Stokes equations have received much attention over past decades due to their importance in the understanding of fluids motion and turbulence. In this paper, we consider the 2D nonautonomous G-Navier-Stokes equations on some bounded domain $\Omega\subset\mathbf{R}^2$ with Dirichlet boundary conditions, which has the following form, (see Roh [1,2] and Jiang and Hou [3])

$$\frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f(x,t),\qquad \text{in } \Omega\times(0,\infty), \tag{1.1a}$$

$$\nabla\cdot(gu)=0,\qquad \text{in } \Omega\times(0,\infty), \tag{1.1b}$$

$$u(x,t)=0,\qquad \text{on } \partial\Omega, \tag{1.1c}$$

$$u(x,0)=u_0(x),\qquad \text{in } \Omega, \tag{1.1d}$$

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where $u(x,t)\in\mathbf{R}^2$ and $p(x,t)\in\mathbf{R}$ denote the velocity and the pressure, $\nu>0$ and $f=f(x,t)\in(L^2(\Omega))^2$ is the time-dependent external force. $0<m_0\leq g=g(x_1,x_2)\leq M_0$. Here, $g=g(x_1,x_2)$ is a suitable real-valued smooth function. When $g=1$, the Eqs. (1.1) become the usual 2D Navier-Stokes equations. In [12], Raugel and Sell proved global existence of strong solutions for large initial data and forcing terms in thin three dimensional domains. In 2005, Roh applied Raugel and Sell methods on $\Omega_g=\Omega_2\times(0,g)$ and derive the 2D G-Navier-Stokes equations form 3D Navier-Stokes equation in [1,2]. In this paper, our aim is to study the long-time behaviour of weak solutions of problem (1.1) by using the theory of pullback attractors. This theory is a natural generalization of the theory of global attractors developed to study autonomous dynamical systems (see [3–12]), and the theory of pullback attractors has an advantage over the theory of uniform attractors (see [13]) allowing the nonautonomous term to be an arbitrary in suitable norms.

Recently, Caraballo in [14] introduces the notion of pullback $\mathcal{D}$-attractor for nonautonomous dynamical systems and prove the existence of pullback $\mathcal{D}$-attractor on some unbounded domains by using the energy equation method. Langa in [15] obtains fractal dimension for 2D N-S equation. Motivated by some ideas in [14,15], we present a new equivalent condition (PC) for pullback $\mathcal{D}$-asymptotically compact by using the measure of noncompactness. In this paper, we prove the existence of pullback attractor and estimate its fractal dimension for 2D G-N-S equation on some bounded domains.

This paper is organized as follows: in Section 1, we recall some basic notations and results for 2D G-Navier-Stokes equations and the concept about the measure of noncompactness. In Section 2, we apply the theory of the measure of noncompactness to obtain the existence of the pullback attractor for non-autonomous G-N-S equation on some bounded domains; then, In Section 3, we estimate the fractal dimension of pullback attractor for 2D G-N-S equation on some bounded domains.

2 Preliminaries

Now, we assume that the Poincaré inequality holds on Ω, there exists an $\lambda_1>0$ such that

$$\int_\Omega \phi^2 g dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 g dx, \quad \forall\phi\in H_0^1(\Omega). \tag{2.1}$$

The mathematical frameworks of (1.1) is the following:

- Let $L^2(g)=(L^2(\Omega))^2$ with the inner products,

$$(u,v)=\int_\Omega u\cdot v g dx \quad \text{and norms} \quad |\cdot|=(\cdot,\cdot)^{\frac{1}{2}}, \quad u,v\in L^2(g).$$

- Let $H_0^1(g)=(H_0^1(\Omega))^2$, which is endowed with the inner products,

$$((u,v))=\int_\Omega\sum_{j=1}^2\nabla u_j\cdot\nabla v_j g dx,$$

and norms

$$\|\cdot\| = ((\cdot,\cdot))^{\frac{1}{2}}\cdot, \qquad u=(u_1,u_2), \qquad v=(v_1,v_2)\in H_0^1(g).$$

Note that thanks to (2.1), the norm $\|\cdot\|$ is equivalent to the usual one in $H_0^1(\Omega)$. Let $D(\Omega)$ be the space of $\mathcal{C}^\infty$ functions with compact support contained in Ω and let

$$\begin{aligned}
&\aleph = \{v\in (D(\Omega))^2 : \nabla\cdot gv = 0 \;\text{ in }\; \Omega\},\\
&H_g = \text{closure of } \;\aleph\; \text{ in } \; L^2(g),\\
&V_g = \text{closure of } \;\aleph\; \text{ in } \; H_0^1(g).
\end{aligned}$$

With H_g and V_g endowed with the inner product and norm of $L^2(g)$ and $H_0^1(g)$ respectively, it follows from (2.1) that

$$|u|^2 \le \frac{1}{\lambda_1}\|u\|^2, \quad \forall u\in V_g. \tag{2.2}$$

Now, we define a G-Laplacian operator as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla\cdot g\nabla)u = -\Delta u - \frac{1}{g}\nabla g\cdot\nabla u,$$

Using the G-Laplacian operator, we rewrite (1.1a) as follows:

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{\nabla g}{g}\cdot\nabla u + (u,\nabla)u + \nabla p = f. \tag{2.3}$$

We define a G-orthogonal projection

$$P_g : L^2(g)\to H_g,$$

and G-Stokes operator

$$A_g u = -P_g\Big(\frac{1}{g}(\nabla\cdot(g\nabla u))\Big),$$

which satisfies the following proposition.

Proposition 2.1. *([1,2]) For the linear operator A_g, the following hold:*

(1) A_g is a positive self-adjoint operator with compact inverse, where the domain of A_g, $D(A_g)=V_g\cap H^2(\Omega)$.

(2) There exist countable eigenvalues of A_g satisfying $0<\lambda_g\le\lambda_1\le\lambda_2\le\lambda_3\le\cdots$, where $\lambda_g=4\pi^2 m_0/M_0$ and λ_1 is the smallest eigenvalue of A_g. In addition, there exists the corresponding collection of eigenfunctions $\{e_1,e_2,e_3,\cdots\}$, which forms an orthonormal basis for H_g.

When we apply the projection P_g into the Eq. (2.3), we can obtain the following weak formulation of (1.1): let $f\in V_g$ and $u_0\in H_g$, we find

$$u \in L^\infty(0,T;H_g) \cap L^2(0,T;V_g), \quad T>0, \tag{2.4}$$

such that

$$\frac{d}{dt}(u,v) + \nu((u,v)) + b_g(u,u,v) + \nu(Ru,v) = \langle f,v\rangle, \quad \forall v \in V_g, \quad \forall t>0, \tag{2.5a}$$

$$u(0) = u_0, \tag{2.5b}$$

where $b_g : V_g \times V_g \times V_g \to \mathbf{R}$ is given by

$$b_g(u,v,w) = \sum_{i,j=1}^{2} \int u_i \frac{\partial v_j}{\partial x} w_j g dx, \tag{2.6a}$$

$$Ru = P_g\Big[\frac{1}{g}(\nabla g \cdot \nabla)u\Big], \quad \forall u \in V_g. \tag{2.6b}$$

Then, the weak formulation (2.5a) is equivalent to the functional equation

$$\frac{du}{dt} + \nu A_g u + Bu + \nu Ru = f, \tag{2.7a}$$

$$u(0) = u_0, \tag{2.7b}$$

where $A_g : V_g \to V_g'$ is the G-Stokes operator defined by

$$\langle A_g u, v\rangle = ((u,v)), \quad \forall u,v \in V_g, \tag{2.8}$$

and $B(u)=B(u,u)=P_g(u\cdot\nabla)u$ is a bilinear operator $B : V_g \times V_g \to V_g'$ defined by

$$\langle B(u,v),w\rangle = b_g(u,v,w), \quad \forall u,v,w \in V_g.$$

Now, we recall some well known inequalities (see Temam [16]) that we will be using in what follows

$$|B(u,v,w)| \le C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}, \quad \forall u,v,w \in V_g, \tag{2.9}$$

here C denote positive constants, which may be different from line to line and even in the same line.

The G-Stokes operator A_g is an isomorphism from V_g into V_g', while B and R satisfy the following inequalities (see Roh [2] and Sell and You [17]):

$$\|B(u)\|_{V_g'} \le c|u|\|u\|, \quad \|Ru\|_{V_g'} \le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\|u\|, \quad \forall u \in V_g. \tag{2.10}$$

We have the following concept and result (see Bae [18] and Temam [16]).

Proposition 2.2. *Given* $f\in L^2(g)$, $u_0(x)\in H_g$, *there exists a unique*

$$u(x,t) \in L^\infty(\mathbf{R}^+;H_g)\cap L^2(0,T;V_g)\cap C(\mathbf{R}^+;H_g), \quad \forall T>0,$$

such that (2.5a)-(2.5b) *hold.*

Now, we recall some basic notions and result about existence of pullback attractors.

Let X be a complete metric space with distance $d(\cdot,\cdot)$. A two-parameter family of mappings acting on X: $U(t,\tau): X\to X$, $t\ge\tau$, $\tau\in\mathbf{R}$, is said to be an evolutionary process if

(1) $U(t,\tau)=U(t,r)U(r,\tau)$, for all $\tau\le r\le t$,

(2) $U(t,\tau)=Id$ is the identity operator, $\tau\in\mathbf{R}$.

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widehat{D}=\{D(t)\colon t\in\mathbf{R}\}\subset\mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X. The following two definitions can be found in [20].

Definition 2.1. *It is said that* $\widehat{B}\in\mathcal{D}$ *is pullback* $\mathcal{D}$*-absorbing for the process* $U(t,\tau)$ *if for any* $t\in\mathbf{R}$ *and any* $\widehat{D}\in\mathcal{D}$, *there exists a* $\tau_0(t,\widehat{D})\le t$ *such that*

$$U(t,\tau)D(\tau)\subset B(t), \quad \text{for } \ \tau\le\tau_0(t,\widehat{D}).$$

Definition 2.2. *A family*

$$\widehat{A}=\{A(t): t\in\mathbf{R}\}\subset\mathcal{P}(X),$$

is said to be a pullback $\mathcal{D}$*-attractor for the process* $U(\cdot,\cdot)$ *in* X *if*

(1) $A(t)$ *is compact for every* $t\in\mathbf{R}$,

(2) $\widehat{A}$ *is pullback* $\mathcal{D}$*-attracting, i.e.,*

$$\lim_{\tau\to-\infty} dist(U(t,\tau)D(\tau),A(t))=0, \quad \text{for all } \ \widehat{D}\in D \ \text{ and all } \ t\in\mathbf{R},$$

(3) $\widehat{A}$ *is invariant, i.e.,* $U(t,\tau)A(\tau)=A(t)$, *for* $-\infty<\tau\le t<+\infty$, *where* $dist(A,B)$ *is the Hausdorff semi-distance between* A *and* B, *defined as*

$$dist(A,B)=\sup_{x\in A}\inf_{y\in B} d(x,y), \quad \text{for } \ A,B\subset X.$$

We call $\widehat{A}$ minimal if for every family $\widehat{C}=\{C(t)\colon t\in\mathbf{R}\subset\mathcal{P}(X)\}$ of closed sets satisfying

$$\lim_{\tau\to-\infty} dist(U(t,\tau)B(\tau),C(\tau))=0,$$

where $A(t)\subset C(t)$.

Let $B(X)$ is the set of all bounded subsets of X and $B\in B(X)$. Its Kuratowski measure of noncompactness $\alpha(B)$ is defined by

$$\alpha(B)=\inf\big\{\delta | B \text{ admits a finite cover by set of diameter } \le\delta\big\}.$$

It has the following properties (see Sell and You [17], Hale [19]).

Lemma 2.1. *Let $B, B_1, B_2 \in B(X)$. Then*

(1) $\alpha(B) = 0 \Leftrightarrow \alpha(N(B,\varepsilon)) \le 2\varepsilon \Leftrightarrow \overline{B}$ is compact;

(2) $\alpha(B_1 + B_2) \le \alpha(B_1) + \alpha(B_2)$;

(3) $\alpha(B_1) \le \alpha(B_2)$ whenever $B_1 \subset B_2$;

(4) $\alpha(B_1 \cup B_2) \le \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) $\alpha(\overline{B}) = \alpha(B)$;

(6) if B is a ball of radius ε, then $\alpha(B) \le 2\varepsilon$.

Lemma 2.2. *Let $\cdots \supset F_n \supset F_{n+1} \supset \cdots$ be a sequence of nonempty closed subsets of X such that $\alpha(F_n) \to 0$, as $n \to \infty$. Then $F = \cap_{n=1}^{\infty} F_n$ is nonempty and compact.*

Definition 2.3. *A process $U(t,\tau)$ on X is said to be pullback $\mathcal{D}$-limit-set compact if for any $\widehat{D} \in \mathcal{D}$,*

$$\lim_{s \to -\infty} \alpha\Big(\bigcup_{\tau < s} U(t,\tau) D(\tau) \Big) = 0.$$

Definition 2.4. *Let X be a Banach space. A process $U(t,\tau)$ is said to be norm-to-weak continuous on X if for all $t, \tau \in \mathbf{R}$ with $t \ge \tau$ and for every sequence $x_n \in X$,*

$$x_n \to x \ \text{ strongly in } \ X \Rightarrow U(t,\tau)x_n \to U(t,\tau)x \ \text{ weakly in } \ X.$$

The following result is very useful to check that the process is norm-to-weak continuous.

Theorem 2.1. *([11]) Let X, Y be two Banach space, X^*, Y^* be respectively their dual spaces. Assume that X is dense in Y, the injection $i : X \to Y$ is continuous, its adjoint $i^* : Y^* \to X^*$ is dense, and U is a norm-to-weak continuous process on Y. Then U is a norm-to-weak continuous process on X if and only if for any $\tau \in \mathbf{R}$, $t \ge \tau$, $U(t,\tau)$ maps compact sets of X to bounded sets of X.*

Theorem 2.2. *([20]) Let X be a Banach space, $U(t,\tau)$ be a norm-to-weak continuous process in X satisfying the following conditions:*

(1) There exists a family $\hat{B}$ of pullback $\mathcal{D}$-absorbing sets in X,

(2) $U(t,\tau)$ is pullback $\mathcal{D}$-limit-set compact,

then there exists a minimal pullback $\mathcal{D}$-attractor $\hat{A}$ in X given by

$$A(t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) B(\tau)}.$$

Definition 2.5. *([20]) Let X be a Banach space. A process $U(t,\tau)$ on X is said to satisfy pullback $\mathcal{D}$-Condition (C) if for any $t \in \mathbf{R}$, $\hat{D} \in \mathcal{D}$ and $\epsilon > 0$, there exist $\tau_0(\hat{D}, t, \epsilon)$ and a finite dimensional subspace X_1 of X such that*

$$P\Big(\bigcup_{\tau < \tau_0} U(t,\tau) D(\tau) \Big) \text{ is bounded}; \tag{2.11a}$$

$$\Big\| (I - P)\Big(\bigcup_{\tau \le \tau_0} U(t,\tau) D(\tau) \Big) \Big\| \le \varepsilon, \tag{2.11b}$$

where $P : X \to X_1$ is a bounded projector.

Theorem 2.3. *([20]) Let $U(t,\tau)$ be a process in a uniformly convex Banach space X. Then the following conditions are equivalent:*

(1) $U(t,\tau)$ satisfies pullback $\mathcal{D}$-Condition (C),

(2) $U(t,\tau)$ is pullback $\mathcal{D}$-limit-set compact.

3 The existence of pullback attractor on bounded domains

Denote by $L^2_{loc}(\mathbf{R},X)$ the metrizable space of function $f(s)$, $s\in\mathbf{R}$ with value in X that are locally 2-power integrable in the Bochner sense , It is equipped with the local 2-power mean convergence topology. Now, we apply the measure of noncompactness to prove the existence of pullback attractor for 2D G-Navier-Stokes equations. We have the following lemma (see Jiang and Hou [21]).

Lemma 3.1. *Suppose $f\in L^2_{loc}(\mathbf{R},H_g)$ is such that*

$$|f|_b^2 = \sup_{t\in\mathbf{R}} \int_t^{t+1} |f(s)|^2 ds < \infty, \tag{3.1}$$

$u_0(x)\in H_g$, and let

$$u(x,t) \in L^\infty(\mathbf{R}^+,H_g) \cap L^2_{loc}(0,T,V_g) \cap \mathcal{C}(\mathbf{R}^+,H_g), \quad \forall t>0,$$

be a weak solution of (1.1)*. Then for all $t\geq\tau$, with $\sigma=\nu\lambda_1$, the following estimate holds,*

$$|u(t)|^2 \leq |u_0|^2 e^{-\sigma\gamma_0(t-\tau)} + R_1^2, \tag{3.2}$$

where

$$R_1^2 = \sigma^{-1}(1-e^{-\sigma\gamma_0})^{-1}|f|_b^2 \quad \textit{and} \quad \gamma_0 = 1 - 2\nu\frac{|\nabla g|_\infty}{m_0\sqrt{\lambda_1}}, \tag{3.3}$$

for sufficiently small $|\nabla g|_\infty$.

For any $f\in L^2_{loc}(\mathbf{R},H_g)$, $|f|_b^2=|f_0|_b^2$, using (3.1), we obtain

$$B_0 = \big\{u \in H_g \big| |u|^2 \leq 2R_1^2 = \rho_0^2\big\}, \tag{3.4}$$

is the pullback $\mathcal{D}$-absorbing set in H_g.

Lemma 3.2. *Suppose $f\in L^2_{loc}(\mathbf{R},H_g)$ satisfying* (3.1) *and $u_0(x)\in H_g$. Let*

$$u(x,t) \in L^\infty(\mathbf{R}^+,V_g) \cap L^2_{loc}(0,T,D(A_g)) \cap \mathcal{C}(\mathbf{R}^+,V_g), \quad u'(x,t) \in L^2_{loc}(\mathbf{R}_\tau;H_g), \ \forall t>0,$$

be a strong solution of (1.1)*, then for all $t\geq\tau$, the following estimates hold:*

$$\|u(t)\|^2 \leq \|u(\tau)\|^2 e^{-\beta(t-\tau)} + (1-e^{-\beta})^{-1}|f|_b^2, \tag{3.5a}$$

where

$$\beta=\lambda\Big(2\nu-1-\frac{2C\rho_0}{\lambda_0^{1/2}}-\frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}}\Big), \tag{3.5b}$$

for sufficiently small $|\nabla g|_\infty$.

Let

$$B_1=\bigcup_{f\in\Gamma}\bigcup_{t>t_0+1}\phi(t_0+1,f,B_0)$$

. By using (3.2), B_1 is bound, $\|u\|^2\leq\rho_1^2$, $\forall u\in B_1$, and B_1 is the pullback $\mathcal{D}$-absorbing set in V_g.

Theorem 3.1. *If* $f(x,t)\in L^2_{loc}(\mathbf{R};H_g)$ *satisfies* (3.1), *then the process* $\{U(t,\tau)\}$ *corresponding to problem* (1.1) *possesses a minimal pullback* $\mathcal{D}$*-attractor*

$$\mathcal{A}=\{A_t\}_{t\in\mathbf{R}}\ \ in\ \ H_g. \tag{3.6}$$

Proof. From Lemma 3.1, we know that the process $U(t,\tau)$ corresponding to problem (1.1) process a family of $\hat{B}$ of pullback $\mathcal{D}$-absorbing sets in H_g. It is easy to see that the process $U(t,\tau)$ is weakly continuous in V_g (see [13]), From Lemma 3.1, we know that the process $U(t,\tau)$ maps bounded sets of H_g to bounded sets of H_g for all $\tau\in\mathbf{R}$, $t\geq\tau$. In view of the Theorem 2.1, it is clear that the process $U(t,\tau)$is norm-to-weak continuous in H_g.

From Theorem 2.3, we need only to verify that the family of process $\{U(t,\tau)\}$ satisfies pullback $\mathcal{D}$-Condition (C) in H_g.

For fixed n, let H_1 be the subspace spanned by $\omega_1,\cdots,\omega_n$, H_2 be the orthogonal complement of H_1 in H_g. For any $u\in D(A_g)$, we write

$$u=u_1+u_2,\quad u_1\in H_1,\quad u_2\in H_2,\quad \text{for any}\ \ u\in H_g.$$

Taking the inner product of the equation of (2.7a) with u_2 in H_g, we have

$$\frac{1}{2}\frac{d}{dt}|u_2|^2+\nu\|u_2\|^2+(B(u),u_2)+\nu(Ru,u_2)=(f,u_2). \tag{3.7}$$

Using Young's inequality, together with (2.9), we have

$$\begin{aligned}|(B(u),u_2)|\ &\leq C|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}|u_2|^{\frac{1}{2}}\|u_2\|^{\frac{1}{2}}\leq C|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}\tfrac{1}{\lambda_1^{1/4}}\|u_2\|\\ &\leq\tfrac{C}{\lambda_1^{1/4}}\Big(\tfrac{\nu\lambda_1^{\frac{1}{4}}\|u_2\|^2}{3C}+\tfrac{3C|u|\|u\|^3}{4\nu\lambda_1^{1/4}}\Big)=\tfrac{\nu}{3}\|u_2\|^2+\tfrac{3C}{4\nu\lambda_1^{1/2}}|u|\|u\|^3,\end{aligned} \tag{3.8}$$

$$\begin{aligned}|(Ru,u_2)|\ &\leq|Ru|\cdot|u_2|\leq\tfrac{|\nabla g|_\infty}{m_0}\|u\|\cdot|u_2|\leq\tfrac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|\cdot\|u_2\|\\ &\leq\tfrac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big(\tfrac{\|u_2\|^2}{3}\cdot\tfrac{m_0\lambda_1^{1/2}}{|\nabla g|_\infty}+\tfrac{3|\nabla g|_\infty}{4m_0\lambda_1^{1/2}}\|u\|^2\Big)\\ &=\tfrac{\nu}{3}\|u_2\|^2+\tfrac{3\nu}{4}\Big(\tfrac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^2\|u\|^2.\end{aligned} \tag{3.9}$$

Moreover,we have

$$2|(f,u_2)| \le 2|f|\cdot|u_2| \le \frac{3|f|^2}{\nu\lambda_1} + \frac{\nu\lambda_1|u_2|^2}{3} \le \frac{3|f|^2}{\nu\lambda_1} + \frac{\nu\|u_2\|^2}{3},$$

and

$$\begin{aligned}
&\frac{d}{dt}|u_2|^2 + 2\nu\|u_2\|^2 = 2(f,u_2) - 2(B(u),u_2) - 2\nu(Ru,u_2)\\
&\le 2|(f,u_2)| + 2|(B(u),u_2)| + 2\nu|(Ru,u_2)|\\
&\le \frac{3|f|^2}{\nu\lambda_1} + \frac{\nu\|u_2\|^2}{3} + \frac{2\nu\|u_2\|^2}{3} + \frac{3C|u|\|u\|^3}{2\nu\lambda_1^{1/2}} + \frac{2\nu\|u_2\|^2}{3} + \frac{3\nu}{2}\Big(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^2\|u\|^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{d}{dt}|u_2|^2 + \frac{\nu}{3}\|u_2\|^2 \le& \frac{3|f|^2}{\nu\lambda_1} + \frac{3C}{2\nu\lambda_1^{1/2}}|u|\|u\|^3 + \frac{3\nu}{2}\Big(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^2\|u\|^2\\
\le& \frac{3|f|^2}{\nu\lambda_1} + \frac{3C}{2\nu\lambda_1^{1/2}}\rho_0\cdot\rho_1^3 + \frac{3\nu}{2}\Big(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^2\rho_1^2.
\end{aligned}$$

Letting

$$m = \frac{3C}{2\nu\lambda_1^{1/2}}\rho_0\cdot\rho_1^3 + \frac{3\nu}{2}\Big(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^2\rho_1^2.$$

we have

$$\frac{d}{dt}|u_2|^2 + \frac{\nu}{3}\|u_2\|^2 \le \frac{3|f|^2}{\nu\lambda_1} + m. \tag{3.10}$$

Therefore

$$\frac{d}{dt}|u_2|^2 + \frac{\nu\lambda_1}{3}|u_2|^2 \le \frac{3|f|^2}{\nu\lambda_1} + m. \tag{3.11}$$

By the Gronwall inequality, we obtain

$$\begin{aligned}
|u_2(t)|^2 \le& |u_2(t_0+1)|^2 e^{-\frac{\nu\lambda_1}{3}(t-(t_0+1))} + \int_{t_0+1}^t e^{-\frac{\nu\lambda_1}{3}(t-s)}\Big(\frac{3|f|^2}{\nu\lambda_1} + m\Big)ds\\
\le& |u_2(t_0+1)|^2 e^{-\frac{\nu\lambda_1}{3}(t-(t_0+1))} + \frac{3m}{\nu\lambda_1}e^{-\frac{\nu\lambda_1}{3}(t_0+1)} + \int_{t_0+1}^t e^{-\frac{\nu\lambda_1}{3}(t-s)}\frac{3|f|^2}{\nu\lambda_1}ds.
\end{aligned}$$

By (3.2) and Lemma 3.1, for any $\epsilon>0$, we can take t large enough such that

$$\int_{t_0+1}^t e^{-\frac{\nu\lambda_1}{3}(t-s)}\frac{3|f|^2}{\nu\lambda_1}ds \le \frac{\epsilon}{3}, \qquad \frac{3m}{\nu\lambda_1}e^{-\frac{\nu\lambda_1}{3}(t_0+1)} \le \frac{\epsilon}{3}. \tag{3.12}$$

Let

$$t_2 = t_0 + 1 + \frac{3}{\nu\lambda_1}\ln\frac{3\rho_0^2}{\epsilon}.$$

Then for $t \geq t_2$, we have

$$|u_2(t_0+1)|^2 e^{-\frac{\nu\lambda_1}{3}(t-(t_0+1))} \leq \rho_0^2 e^{-\frac{\nu\lambda_1}{3}(t-(t_0+1))} \leq \frac{\epsilon}{3}.$$

Hence we have

$$|u_2(t)|^2 \leq \epsilon, \quad \forall t \geq t_2,$$

which indicates that the process $\{U(t,\tau)\}$ in H_g satisfies pullback condition (PC) in H_g. Applying Theorems 2.10, the proof is completed. □

4 The dimension of pullback attractors in H_g

Let $F : V_g \times \mathbf{R} \to V_g'$ be a given family of nonlinear operators such that, for all $\tau \in \mathbf{R}$, and any $u_0 \in H_g$, there exists a unique function $u(t) = u(t;\tau,u_0)$ satisfying

$$u \in L^2(\tau,T;V_g) \cap C([\tau,T];H_g), \quad F(u(t),t) \in L'(\tau,T;V_g'), \quad \text{for all } T > \tau, \tag{4.1a}$$

$$\frac{du}{dt} = F(u(t),\tau), \quad t > \tau, \tag{4.1b}$$

$$u(\tau) = u_0. \tag{4.1c}$$

We define

$$U(t,\tau)u_0 = u(t;\tau,u_0), \quad \tau \leq t, \quad u_0 \in H_g.$$

Let $T^* \in \mathbf{R}$ be fixed, we assume that there exists a family $\{K(t); t \leq T^*\}$ of non-empty compact subsets of H_g satisfying the invariance property

$$U(t,\tau)K(\tau) = K(t), \quad \text{for all } \tau \leq t \leq T^*. \tag{4.2}$$

We have

Lemma 4.1. *([15]) Let us suppose that*

$$\bigcup_{\tau \leq T^*} K(\tau) \text{ is relatively compact in } H_g,$$

and there exist q_j, $j = 1,2,\cdots$, such that

$$\tilde{q}_j \leq q_j, \text{ for any } j \geq 1, \tag{4.3a}$$

$$q_{n_0} \geq 0, \quad q_{n_0+1} < 0 \text{ for some } n_0 \geq 1, \tag{4.3b}$$

$$q_j \leq q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - j), \text{ for all } j = 1,2,\cdots. \tag{4.3c}$$

Then

$$d_F(K(\tau)) \leq d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}}, \quad \text{for all } \tau \leq T^*, \tag{4.4a}$$

where

$$\tilde{q}_j = \limsup_{T\to+\infty} \sup_{\tau\leq T^*} \sup_{u_0\in K(\tau-T)} \Big(\frac{1}{T}\int_{\tau-T}^{\tau} Tr_j(F'(U(s,\tau-T)u_0,s))ds\Big), \tag{4.4b}$$

$$F' : (u,t) \in V_g \times (-\infty, T^*] \to F'(u,t) \in \mathcal{L}(V_g, V_g'). \tag{4.4c}$$

Below we will give the following main result.

Theorem 4.1. *Suppose that $f\in L^2_{loc}(\mathbf{R}; V_g')$ is such that*

$$\int_{-\infty}^{t} \|f(s)\|_*^2 ds < +\infty, \quad \textit{for all } \ t \in \mathbf{R}.$$

Then, the dimension of pullback attractor satisfies

$$d_F(A(\tau)) \leq \max\Big(1, \frac{c}{\lambda_1 \nu^4 \tilde{m}^2}\|f\|^2_{L^\infty(-\infty,T^*;V_g')}\Big), \quad \textit{for all } \ \tau \in \mathbf{R},$$

where $\tilde{m} = 1 - 2|\nabla g|_\infty / m_0\lambda_1^{1/2}$ for sufficiently small $|\nabla g|_\infty$.

Proof. Since problems (2.5a) and (2.5b) can be written in the form (4.1b) and (4.1c), we let

$$F(u,t) = -\nu Au - B(u) - \nu Ru + f(t).$$

For each $t\leq T^*$, the mapping $F(\cdot,t)$ is Fréchet differentiable in V_g, and $u,v \in V_g$ is continuous,

$$F'(u,t)v = -\nu Av - B(u,v) - B(v,u) - \nu(Ru,v), \tag{4.5}$$

Let $u_0, v_0^1, \cdots, v_0^j \in H_g$, and $\tau\leq T^*$ be fixed. Let $\varphi_1(s), \varphi_2(s), \cdots, \varphi_j(s), s\geq\tau$, be an orthonomal basis in H_g of the subspace spanned by $v(s;\tau,u_0,v_0'), \cdots, v(s;\tau,u_0,v_0^j)$, the corresponding solution of (4.1b). We can assume that $\varphi_i(s)\in V_g$ almost everywhere $s\geq\tau$ with

$$Tr_j(F'(U(s,\tau)u_0,s)) = \sup_{\substack{v_0^i \in H_g \\ |v_0^i| \leq 1, i \leq j}} \Big(\sum_{i=1}^{j}(F'(U(s,\tau)u_0,s)\varphi_i,\varphi_i\Big). \tag{4.6}$$

Note that

$$\sum_{i=1}^{j}(F'(U(s,\tau)u_0,s)\varphi_i,\varphi_i) = \sum_{i=1}^{j}\big(-\nu A\varphi_i - B(\varphi_i, U(s,\tau)u_0,\phi_i) - \nu R(\varphi_i,\varphi_i)\big).$$

We also have

$$
\begin{aligned}
\Big|\sum_{i=1}^{j} B(\varphi_i, U(s,\tau)u_0, \varphi_i)\Big| =& \Big|\int_\Omega \sum_{i=1}^{j}\sum_{k,l=1}^{2} \varphi_{ik} D_k(U(s,\tau)u_0)_l(x)\varphi_{il}(x)g(x)dx\Big| \\
\le& \int_\Omega |grad(U(s,\tau)u_0)(x)\rho(x)|dx \\
\le& \|U(s,\tau)\||\rho| \quad \text{(with the Schwarz inequality)},
\end{aligned}
$$

where

$$
\rho(x) = \sum_{i}^{j} |\sqrt{g}\varphi_i(x)|^2.
$$

We have the Lieb-Thirring inequality

$$
|\rho(\tau)|^2 = \int_\Omega \rho^2(x,\tau)g(x)dx \le c\sum_{i=1}^{j} \|\varphi_i\|^2, \tag{4.7a}
$$

$$
\Big|\sum_{i=1}^{j} \nu(R\varphi_i, \varphi_i)\Big| = \Big|\sum_{i=1}^{j} \Big(\frac{\nu}{g}(\nabla g \cdot \nabla \varphi_i)\varphi_i\Big)\Big| \le \sum_{i=1}^{j} \frac{\nu|\nabla g|_\infty}{m_0}\|\varphi_i\|\|\varphi_i\|. \tag{4.7b}
$$

Consequently,

$$
\begin{aligned}
& \textstyle\sum_{i=1}^{j} \big(F'(U(s,\tau)u_0, s)\varphi_i, \varphi_i\big) \\
\le & -\nu \textstyle\sum_{i=1}^{j} \|\varphi_i\|^2 + \|U(s,\tau)u_0\|\rho + \sum_{i=1}^{j} \nu \frac{|\nabla g|_\infty}{m_0}\|\varphi_i\|\|\varphi_i\| \\
\le & -\frac{\nu}{2}\textstyle\sum_{i=1}^{j} \|\varphi_i\|^2 + \frac{c}{2\nu}\|U(s,\tau)u_0\|^2 + \sum_{i=1}^{j} \nu \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|\varphi_i\|^2 \\
\le & -\frac{\nu\lambda_1}{2}\Big(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)\textstyle\sum_{i=1}^{j} |\varphi_i|^2 + \frac{c}{2\nu}\|U(s,\tau)u_0\|^2 \\
= & -\frac{\sigma}{2}\Big(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)j + \frac{c}{2\nu}\|U(s,\tau)u_0\|^2;
\end{aligned} \tag{4.8}
$$

where in the second step above, we have used the Schwarz inequality and (3.4). Therefore, we have

$$
Tr_j(F'(U(s,\tau)u_0, s) \le -\frac{\sigma}{2}\Big(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)j + \frac{c}{2\nu}\|U(s,\tau)u_0\|^2. \tag{4.9}
$$

On the other hand,

$$
\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 = 2(f,u) - 2\nu\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)u, u\Big),
$$

which gives

$$\frac{d}{dt}|u|^2+2\nu\|u\|^2\le 2(f,u)-2\nu\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)u,u\Big),$$

and

$$\begin{aligned}&|U(t,\tau)u_0|^2+2\nu\int_\tau^t\|U(s,\tau)u_0\|^2ds\\&\le|u_0|^2+2\int_\tau^t(f(s),U(s,\tau)u_0)ds-2\nu\int_\tau^t\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)U(s,\tau)u_0,U(s,\tau)u_0\Big)ds.\end{aligned}$$

Since

$$2\int_\tau^t(f(s),U(s,\tau)u_0)ds\le\nu\int_\tau^t\|U(s,\tau)u_0\|^2ds+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds,$$

we have

$$\begin{aligned}\nu\int_\tau^t\|U(s,\tau)u_0\|^2ds\le&|u_0|^2+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds-2\nu\int_\tau^t\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)U(s,\tau)u_0,U(s,\tau)u_0\Big)ds\\\le&|u_0|^2+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds+\frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\int_\tau^t\|U(s,\tau)u_0\|^2ds,\end{aligned}$$

which yields

$$\Big(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)\nu\int_\tau^t\|U(s,\tau)u_0\|^2ds\le|u_0|^2+\frac{1}{\nu}\int_\tau^t\|f(s)\|_*^2ds. \tag{4.10}$$

By letting

$$\tilde{m}=1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}},$$

we have

$$\int_\tau^t\|U(s,\tau)u_0\|^2ds\le\frac{|u_0|^2}{\tilde{m}\nu}+\frac{1}{\tilde{m}\nu^2}\int_\tau^t\|f(s)\|_*^2ds. \tag{4.11}$$

Let $M=\|f\|^2_{L^\infty(-\infty,T^*;V_g')}$. We have

$$\tilde{q}_j=-\frac{\sigma\tilde{m}}{2}j+\frac{c}{2\tilde{m}\nu^3}\limsup_{T\to+\infty}\frac{1}{T}\int_{\tau-T}^\tau\|f(s)\|_*^2ds\le-\frac{\sigma\tilde{m}}{2}j+\frac{cM}{2\tilde{m}\nu^3}.$$

Hence if $M<\sigma\tilde{m}^2\nu^3/c$, we take

$$q_j=-\frac{\sigma\tilde{m}}{2}(j-1),\quad j=1,2,\cdots,\quad\text{and}\quad n_0=1,$$

we can obtain

$$d_F(A_\sigma(t))\le 1,\quad\text{for all }\ \tau\le T^*. \tag{4.12}$$

If $M>\sigma\tilde{m}^2\nu^3/c$, then taking

$$q_j=-\frac{\sigma\tilde{m}}{2}j+\frac{cM}{2\tilde{m}\nu^3},\quad j=1,2,\cdots,\quad n_0=1+\Big[\frac{cM}{\sigma\tilde{m}^2\nu^3}-1\Big],$$

where $[\cdot]$ denotes the integer part of a real number, we can obtain by using Lemma 4.1 that

$$d_F(A_\sigma(t))\le\frac{cM}{\sigma\tilde{m}^2\nu^3}=cM\Big[\lambda_1\nu^4\Big(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^2\Big]^{-1},\quad \text{for all}\ \ \tau\le T^*. \tag{4.13}$$

Since

$$\bigcup_{t\le T^*}A(\tau)\ \text{is relatively compact in}\ \ H_g,$$

whose proof we omit as it is similar to Theorem 3.6 in [15]. we obtain from (4.12) and (4.13) that,

$$d_F(A(\tau))\le\max\Big(1,\frac{c}{\lambda_1\nu^4\tilde{m}^2}\|f\|^2_{L^\infty(-\infty,T^*;V_g')}\Big),\quad \text{for all}\ \ \tau\in\mathbf{R}.$$

Then theorem is proved. □

Acknowledgments

The author would like to thank the referees for the helpful suggestions. This work was partially supported by the National Natural Science Fund of China (Grant No. 10871156) and the Fund of XJTU (Grant No. 2009xjtujc30).

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