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Exact Vibration Solutions of Nonhomogeneous Circular, Annular and Sector Membranes

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Abstract. In this paper, exact vibration frequencies of circular, annular and sector membranes with a radial power law density are presented for the first time. It is found that in general, the sequence of modes may not correspond to increasing azimuthal mode number n. The normalized frequency increases with the absolute value of the power index $|\nu|$. For a circular membrane, the fundamental frequency occurs at $n = 0$ where n is the number of nodal diameters. For an annular membrane, the frequency increases with respect to the inner radius b. When b is close to one, the width $1 - b$ is the dominant factor and the differences in frequencies are small. For a sector membrane, $n - 1$ is the number of internal radial nodes and the fundamental frequency occurs at $n = 1$. Increased opening angle β increases the frequency.

AMS subject classifications: 74.K15, 74.H45

Key words: Membrane, vibration, non-homogeneous, exact, circular, annular, sector

1 Introduction

The solution to the Helmholtz equation governing vibrating membranes is important in the design of drums, speakers, receivers and electromagnetic waveguides. The membrane with a uniform density (or thickness), or joined uniform pieces, has been well researched and documented. In contrast, literature on continuous, nonhomogeneous membrane are rather few, especially those that gave exact solutions. Exact solutions are useful for checking approximate results from numerical or series solutions.

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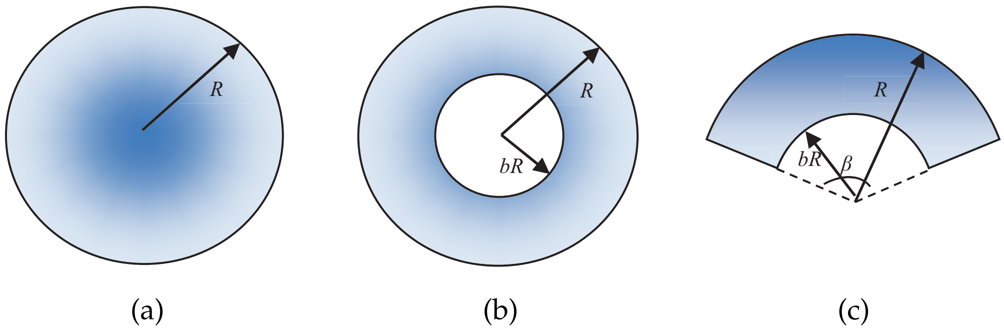

Figure 1: (a) Circular membrane. (b) Annular membrane. (c) Annular sector membrane.

For rectangular membranes, the only solutions seem to be that of Wang [1], who considered a membrane with a linear taper and Wang and Wang [2] who treated membranes with linear density distribution and exponential density distribution. For an annular membrane, De [3] and Wang [1] studied membranes with a density that varies as inverse radius squared, while Gottlieb [4] gave a solution with inverse radius to the fourth power. We mention some papers which considered axisymmetric vibrations only [5,6]. Such solutions would not give the complete frequency spectrum, since the asymmetric modes are not included.

This paper presents some new exact solutions for the circular or annular membranes and sector membranes whose densities are functions of the radius (see Figs. 1(a) to 1(c)).

2 Problem definition

Consider a membrane with radius R, under uniform stress T_0 and maximum density ρ_0. The normalized membrane equation of motion, in cylindrical coordinates (r,θ) is given by

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\omega^2\rho(r)w=0, \tag{2.1}$$

where w is the transverse deflection, ρ the mass density which is a function of radius and normalized by ρ_0, and ω the circular frequency normalized by $\sqrt{T_0/\rho_0 R^2}$. The boundary conditions are that $w=0$ on the boundaries. The problem at hand is to determine the exact circular frequencies of the aforementioned nonhomogeneous membranes.

3 Power law density distribution

We first consider the normalized density (or thickness) that is described by the following power law

$$\rho = cr^{\nu}, \tag{3.1}$$

Table 1: Frequencies for full circular membrane with power law density. The azimuthal mode n is in parentheses.

$\nu = 0$	0.5	1.0	1.5	2.0	3.0
2.4048 (0)	3.0060 (0)	3.6072 (0)	4.2084 (0)	4.8097 (0)	6.0121 (0)
3.8317 (1)	4.4497 (1)	5.0634 (1)	5.6743 (1)	6.2836 (1)	7.4971 (1)
5.1356 (2)	5.7790 (2)	6.4130 (2)	7.0406 (2)	7.6634 (2)	8.8995 (2)
5.5201 (0)	6.9001 (0)	7.7034 (3)	8.3487 (3)	8.9868 (3)	10.248 (3)
6.3802 (3)	7.0486 (3)	8.2801 (0)	9.6178 (4)	10.271 (4)	11.558 (4)

where ν is a constant exponent, and $c = 1$ if $\nu \geq 0$. If $\nu < 0$, the density is unbounded at the origin, and only the annular membrane is appropriate. In that case let the inner radius of the annular membrane be bR, where the maximum density occurs and set $c = b^{|\nu|}$.

For full membranes (Fig. 1(a)), let the solution of Eq. (2.1) be

$$w = \cos(n\theta) f(r), \tag{3.2}$$

where n is an integer. In view of this assumed solution, Eq. (2.1) becomes

$$r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} - n^2 f + \omega^2 c r^{\nu+2} f = 0. \tag{3.3}$$

By letting

$$\gamma = \frac{\nu}{2} + 1, \tag{3.4}$$

the solution to Eq. (3.3) is given by

$$f = C_1 J_{\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma} r^{\gamma}\Big) + C_2 J_{-\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma} r^{\gamma}\Big), \tag{3.5}$$

provided $\gamma \neq 0$. Here J is the Bessel function of the first kind. If n/γ is an integer, the second solution is replaced by the Bessel function Y.

For a full circular membrane, $\nu \geq 0$ and $c = 1$. The boundary conditions at $r = 1$ gives $C_2 = 0$ and the frequency equation is thus given by

$$J_{\frac{n}{\gamma}}\Big(\frac{\omega}{\gamma}\Big) = 0. \tag{3.6}$$

Thus (ω/λ) are the zeros of Eq. (3.6). Table 1 shows the results. The $\nu = 0$ case is the uniform membrane, whose frequencies are governed by $J_n(\omega) = 0$. The $\nu = 1$ case is a membrane with a linear density (or thickness).

For an annular membrane, ν can be both positive or negative. The boundary condition at the inner edge gives

$$f = C_1\Big[J_{-\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma} b^{\gamma}\Big) J_{\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma} r^{\gamma}\Big) - J_{\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma} b^{\gamma}\Big) J_{-\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma} r^{\gamma}\Big)\Big]. \tag{3.7}$$

The boundary condition at the outer edge then gives the characteristic equation

$$\Big[J_{-\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma}b^{\gamma}\Big)J_{\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma}\Big)-J_{\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma}b^{\gamma}\Big)J_{-\frac{n}{\gamma}}\Big(\frac{\omega\sqrt{c}}{\gamma}\Big)\Big]=0. \tag{3.8}$$

If $\nu=-2$, $\gamma=0$. In this case, Eq. (3.3) degenerates to

$$r^2\frac{d^2f}{dr^2}+r\frac{df}{dr}-n^2f+\omega^2b^2f=0. \tag{3.9}$$

The solution was found by [1,2]

$$f=C\sin\big(\sqrt{\omega^2b^2-n^2}\ln r\big). \tag{3.10}$$

By setting $f=0$ at $r=b$, the frequencies are found to be in a closed form given by

$$\omega=\frac{1}{b}\Big[\Big(\frac{m\pi}{\ln b}\Big)^2+n^2\Big]^{\frac{1}{2}}. \tag{3.11}$$

Here m is a positive integer. Some results for the annular membrane are given in Table 2.

Table 2: Normalized frequencies ω for annular membrane with power index ν. The azimuthal mode number n is in parenthesis.

b	$\nu=-2$	$\nu=-1$	$\nu=0$	$\nu=1$	$\nu=2$
0.1	13.644 (0)	7.1519 (0)	3.3139 (0)	4.4410 (0)	5.6018 (0)
	16.916 (1)	8.7465 (1)	3.9409 (1)	5.1412 (1)	6.3467 (1)
	24.211 (2)	12.075 (2)	5.1424 (2)	6.4165 (2)	7.6658 (2)
	27.288 (0)	14.465 (0)	6.3805 (3)	7.7035 (3)	8.9869 (3)
	29.062 (1)	15.454 (1)	6.8576 (0)	8.9552 (4)	10.271 (4)
0.3	8.6979 (0)	6.3125 (0)	4.4124 (0)	5.4451 (0)	6.5309 (0)
	9.3147 (1)	6.7526 (1)	4.7058 (1)	5.7830 (1)	6.9046 (1)
	10.959 (2)	7.9181 (2)	5.4702 (2)	6.6464 (2)	7.8420 (2)
	13.253 (3)	9.5241 (3)	6.4937 (3)	7.7678 (3)	9.0296 (3)
	15.920 (4)	11.357 (4)	7.6229 (4)	8.9708 (4)	10.280 (4)
0.5	9.0647 (0)	7.5730 (0)	6.2461 (0)	7.7162 (0)	8.1954 (0)
	9.2827 (1)	7.7542 (1)	6.3932 (1)	7.3615 (1)	8.3776 (1)
	9.9080 (2)	8.2734 (2)	6.8138 (2)	7.8327 (2)	8.8950 (2)
	10.871 (3)	9.0715 (3)	7.4577 (3)	8.5495 (3)	9.6763 (3)
	12.090 (4)	10.080 (4)	8.2667 (4)	9.4422 (4)	10.640 (4)
0.7	12.583 (0)	11.490 (0)	10.455 (0)	11.333 (0)	12.243 (0)
	12.664 (1)	11.563 (1)	10.522 (1)	11.405 (1)	12.320 (1)
	12.903 (2)	11.782 (2)	10.720 (2)	11.618 (2)	12.548 (2)
	13.293 (3)	12.137 (3)	11.042 (3)	11.964 (3)	12.918 (3)
	13.820 (4)	12.617 (4)	11.476 (4)	12.431 (4)	13.417 (4)
0.9	33.131 (0)	32.265 (0)	31.412 (0)	32.226 (0)	33.051 (0)
	33.149 (1)	32.283 (1)	31.429 (1)	32.244 (1)	33.069 (1)
	33.205 (2)	32.337 (2)	31.482 (2)	32.298 (2)	33.125 (2)
	33.298 (3)	32.427 (3)	31.570 (3)	32.388 (3)	33.217 (3)
	33.427 (4)	32.554 (4)	31.693 (4)	32.514 (4)	33.346 (4)

Table 3: Frequencies for circular sector. The number n is in parenthesis.

β	$\nu = 0$	$\nu = 0.5$	$\nu = 1.0$	$\nu = 1.5$	$\nu = 2.0$	$\nu = 3.0$
$\pi/4$	7.5883 (1)	8.2803 (1)	8.9552 (1)	9.6178 (1)	10.271 (1)	11.558 (1)
	11.065 (1)	12.531 (1)	13.742 (2)	14.467 (2)	15.177 (2)	16.561 (2)
	12.225 (2)	12.997 (2)	13.977 (1)	15.411 (1)	16.835 (1)	19.662 (1)
	14.373 (1)	16.612 (1)	18.338 (3)	19.115 (3)	19.872 (3)	21.341 (3)
	16.038 (2)	17.535 (3)	18.832 (1)	20.643 (2)	22.129 (2)	25.061 (2)
$\pi/2$	5.1356 (1)	5.7790 (1)	6.4130 (1)	7.0406 (1)	7.6634 (1)	8.8995 (1)
	7.5883 (2)	8.2803 (2)	8.9552 (2)	9.6178 (2)	10.271 (2)	11.558 (2)
	8.4172 (1)	9.8312 (1)	11.236 (1)	12.078 (3)	12.760 (3)	14.097 (3)
	9.9361 (3)	10.671 (3)	11.383 (3)	12.636 (1)	14.031 (1)	16.561 (4)
	11.065 (2)	12.531 (2)	13.742 (4)	14.467 (4)	15.177 (4)	16.814 (1)
π	3.8317 (1)	4.4497 (1)	5.0634 (1)	5.6743 (1)	6.2832 (1)	7.4971 (1)
	5.1356 (2)	5.7790 (2)	6.4130 (2)	7.0406 (2)	7.6634 (2)	8.8995 (2)
	6.3802 (3)	7.0486 (3)	7.7034 (3)	8.3487 (3)	8.9868 (3)	10.248 (3)
	7.0156 (1)	8.2803 (4)	8.9552 (4)	9.6178 (4)	10.271 (4)	11.558 (4)
	7.5883 (4)	8.4071 (1)	9.7954 (1)	10.859 (5)	11.527 (5)	12.839 (5)
2π	3.1416 (1)	3.7486 (1)	4.3539 (1)	4.9582 (1)	5.5618 (1)	6.7677 (1)
	3.8317 (2)	4.4497 (2)	5.0634 (2)	5.6743 (2)	6.2832 (2)	7.4971 (2)
	4.4934 (3)	5.1240 (3)	5.7476 (3)	6.3665 (3)	6.9820 (3)	8.2064 (3)
	5.1356 (4)	5.7790 (4)	6.4130 (4)	7.0406 (4)	7.6634 (4)	8.8995 (4)
	5.7635 (5)	6.4195 (5)	7.0640 (5)	7.7004 (5)	8.3309 (5)	9.5793 (5)

For circular sector or annular sector membranes (Fig. 1(b),1(c)), let the opening angle be β. Let the solution be given by

$$w = \sin(\alpha\theta) f(r), \tag{3.12}$$

where $\alpha = n\pi/\beta$. Note that n is a non-zero positive integer. Then the frequency equation is similar to that for the full membrane, only with n replaced by α. For example, Eq. (3.6) becomes

$$J_{\frac{n\pi}{(\beta\gamma)}}\left(\frac{\omega}{\gamma}\right) = 0. \tag{3.13}$$

Table 3 shows the frequencies for the circular sector.

The substitution is similar for the annular sector. For example, for $\nu = -2$ the formula is given by

$$\omega = \frac{1}{b}\left[\left(\frac{m\pi}{\ln b}\right)^2 + \left(\frac{n\pi}{\beta}\right)^2\right]^{\frac{1}{2}}. \tag{3.14}$$

Owing to the many parameters, we present only some representative cases in Tables 4-7.

4 A special density distribution for annular membrane

The annulus is between $r = b < 1$ and $r = 1$. Let the density distribution be given by

$$\rho = \frac{b^2\left[1 + c\ln\left(\frac{r}{b}\right)\right]}{r^2}. \tag{4.1}$$

Table 4: Frequencies for annular sector membrane, $\beta = \pi/4$.

b	$\nu=-2$	$\nu=-1$	$\nu=0$	$\nu=1$	$\nu=2$
0.25	18.389 (1)	12.311 (1)	7.5984 (1)	8.9592 (1)	10.273 (1)
	24.180 (1)	16.841 (1)	11.169 (1)	13.742 (2)	15.177 (2)
	31.552 (1)	21.086 (2)	12.225 (2)	14.011 (1)	16.849 (1)
	33.259 (2)	21.972 (1)	14.765 (1)	18.338 (3)	19.872 (3)
	39.632 (1)	25.443 (2)	16.038 (2)	18.957 (1)	22.129 (2)
0.5	12.090 (1)	10.080 (1)	8.2667 (1)	9.4422 (1)	10.640 (1)
	18.389 (2)	15.243 (2)	12.311 (2)	13.781 (2)	15.197 (2)
	19.816 (1)	16.583 (1)	13.742 (1)	15.939 (1)	18.289 (1)
	24.180 (2)	20.268 (2)	16.706 (3)	18.340 (3)	19.873 (3)
	25.655 (3)	21.075 (3)	16.843 (2)	19.539 (2)	22.337 (2)
0.75	15.567 (1)	14.413 (1)	13.366 (1)	14.278 (1)	15.218 (1)
	18.050 (2)	16.774 (2)	15.548 (2)	16.593 (2)	17.664 (2)
	21.634 (3)	20.694 (3)	18.616 (3)	19.836 (3)	21.075 (3)
	25.829 (4)	23.986 (4)	22.184 (4)	23.597 (4)	25.007 (4)
	29.605 (1)	27.526 (1)	25.546 (1)	27.326 (1)	29.178 (1)

Table 5: Frequencies for annular sector membrane, $\beta = \pi/2$.

b	$\nu=-2$	$\nu=-1$	$\nu=0$	$\nu=1$	$\nu=2$
0.25	12.090 (1)	8.2667 (1)	5.3199 (1)	6.5327 (1)	7.7518 (1)
	18.389 (2)	12.311 (2)	7.5984 (2)	8.9592 (2)	10.273 (2)
	19.816 (1)	13.742 (1)	9.1444 (1)	11.383 (3)	12.760 (3)
	24.180 (2)	16.706 (3)	9.9365 (3)	11.658 (1)	14.301 (1)
	25.655 (3)	16.841 (2)	11.169 (2)	14.011 (2)	15.177 (4)
0.5	9.9080 (1)	8.2734 (1)	6.8138 (1)	7.8327 (1)	8.8950 (1)
	12.090 (2)	10.080 (2)	8.2667 (2)	9.4422 (2)	10.640 (2)
	15.039 (3)	12.568 (3)	10.189 (3)	11.528 (3)	12.853 (3)
	18.389 (4)	15.243 (4)	12.311 (4)	13.781 (4)	15.197 (4)
	18.566 (1)	15.533 (1)	12.856 (1)	14.893 (1)	17.074 (1)
0.75	14.803 (1)	13.760 (1)	12.761 (1)	13.635 (1)	14.537 (1)
	15.507 (2)	14.413 (2)	13.366 (2)	14.278 (2)	15.218 (2)
	16.614 (3)	15.441 (3)	14.320 (3)	15.287 (3)	16.285 (3)
	18.050 (4)	16.774 (4)	15.547 (4)	16.593 (4)	17.664 (4)
	19.743 (5)	18.345 (5)	16.990 (5)	18.129 (5)	19.281 (5)

For the density to be positive, $c > 1/\ln b$. Using Eq. (3.2), Eq. (2.1) becomes

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} - n^2 f + \omega^2 b^2\Big[1 + c\ln\Big(\frac{r}{b}\Big)\Big]f = 0. \tag{4.2}$$

Let

$$f(r) = g(\eta), \qquad r = e^{\eta-1}, \qquad \eta = 1 + \ln r. \tag{4.3}$$

Eq. (4.3) becomes

$$\frac{d^2g}{d\eta^2} + \big\{\omega^2 b^2\big[1 + c(\eta - 1 - \ln b)\big] - n^2\big\}g = 0. \tag{4.4}$$

Let

$$g(\eta) = h(z), \qquad z = (\omega^2 b^2 c)^{-\frac{2}{3}}\big\{\omega^2 b^2\big[1 + c(\eta - 1 - \ln b)\big] - n^2\big\}. \tag{4.5}$$

Table 6: Frequencies for annular sector membrane, $\beta = \pi$.

b	$\nu = -2$	$\nu = -1$	$\nu = 0$	$\nu = 1$	$\nu = 2$
	9.9080 (1)	6.8138 (1)	4.4475 (1)	5.5557 (1)	6.7021 (1)
	12.090 (2)	8.2667 (2)	5.3199 (2)	6.5327 (2)	7.7518 (2)
0.25	15.039 (3)	10.189 (3)	6.4265 (3)	7.7270 (3)	9.0017 (3)
	18.389 (4)	12.311 (4)	7.5984 (4)	8.9592 (1)	10.273 (4)
	15.566 (1)	12.851 (1)	8.5369 (1)	10.180 (2)	11.527 (5)
	9.2827 (1)	7.7542 (1)	6.3932 (1)	7.3615 (1)	8.3776 (1)
	9.9080 (2)	8.2734 (2)	6.8138 (2)	7.8327 (2)	8.8950 (2)
0.5	10.871 (3)	9.0715 (3)	7.4577 (3)	8.5495 (3)	9.6763 (3)
	12.090 (4)	10.080 (4)	8.2667 (4)	9.4422 (4)	10.640 (4)
	13.497 (5)	11.241 (5)	9.1900 (5)	10.450 (5)	11.715 (5)
	14.621 (1)	13.591 (1)	12.606 (1)	13.470 (1)	14.362 (1)
	14.803 (2)	13.760 (2)	12.790 (2)	13.635 (2)	14.537 (2)
0.75	15.100 (3)	14.036 (3)	13.017 (3)	13.907 (3)	14.825 (3)
	15.507 (4)	14.413 (4)	13.345 (4)	14.278 (4)	15.218 (4)
	16.014 (5)	14.885 (5)	13.784 (5)	14.741 (5)	15.707 (5)

Table 7: Frequencies for annular sector membrane, $\beta = 2\pi$.

b	$\nu = -2$	$\nu = -1$	$\nu = 0$	$\nu = 1$	$\nu = 2$
	9.2827 (1)	6.3932 (1)	4.1888 (1)	5.2583 (1)	6.3755 (1)
	9.9080 (2)	6.8138 (2)	4.4475 (2)	5.5557 (2)	6.7021 (2)
0.25	10.871 (3)	7.4577 (3)	4.8382 (3)	5.9979 (3)	7.1812 (3)
	12.090 (4)	8.2667 (4)	5.3199 (4)	6.5327 (4)	7.7518 (4)
	13.497 (5)	9.1900 (5)	5.8577 (5)	7.1183 (5)	8.3680 (5)
	9.1197 (1)	7.6187 (1)	6.2832 (1)	7.2379 (1)	8.2414 (1)
	9.2827 (2)	7.7542 (2)	6.3932 (2)	7.3615 (2)	8.3776 (2)
0.5	9.5483 (3)	7.9747 (3)	6.5720 (3)	7.5621 (3)	8.5982 (3)
	9.9080 (4)	8.2734 (4)	6.8138 (4)	7.8327 (4)	8.8950 (4)
	10.352 (5)	8.6419 (5)	7.1116 (5)	8.1648 (5)	9.2579 (5)
	14.576 (1)	13.549 (1)	12.566 (1)	13.428 (1)	14.317 (1)
	14.621 (2)	13.591 (2)	12.634 (2)	13.470 (2)	14.362 (2)
0.75	14.697 (3)	13.662 (3)	12.671 (3)	13.539 (3)	14.435 (3)
	14.803 (4)	13.760 (4)	12.713 (4)	13.635 (4)	14.537 (4)
	14.937 (5)	13.885 (5)	12.877 (5)	13.758 (5)	14.667 (5)

Eq. (4.4) becomes the Stokes equation

$$\frac{d^2h}{dz^2} + zh = 0, \tag{4.6}$$

and the solution is

$$h = \sqrt{z}\Big[C_1 J_{-\frac{1}{3}}\Big(\frac{2z^{\frac{3}{2}}}{3}\Big) + C_2 J_{-\frac{1}{3}}\Big(\frac{2z^{\frac{3}{2}}}{3}\Big)\Big]. \tag{4.7}$$

At $r = b$, $\eta = 1 + \ln b$,

$$z = z_0 = (\omega^2 b^2 c)^{-\frac{2}{3}}\big[\omega^2 b^2 - n^2\big]. \tag{4.8}$$

At $r = 1$, $\eta = 1$, and therefore

$$z = z_1 = (\omega^2 b^2 c)^{-\frac{2}{3}}\big[\omega^2 b^2 (1 - c\ln b) - n^2\big]. \tag{4.9}$$

Let

$$p=\frac{2z^{\frac{3}{2}}}{3},\qquad p_0=\frac{2{z_0}^{\frac{3}{2}}}{3},\qquad p_1=\frac{2{z_1}^{\frac{3}{2}}}{3}. \tag{4.10}$$

The solution to Eq. (4.2) is thus

$$f=C_1\sqrt{z}\big[J_{\frac{1}{3}}(p_0)J_{-\frac{1}{3}}(p)-J_{-\frac{1}{3}}(p_0)J_{\frac{1}{3}}(p)\big]. \tag{4.11}$$

The frequency equation is

$$J_{\frac{1}{3}}(p_0)J_{-\frac{1}{3}}(p_1)-J_{-\frac{1}{3}}(p_0)J_{\frac{1}{3}}(p_1)=0. \tag{4.12}$$

For sector membranes, we use Eq. (3.12) instead of Eq. (3.2). All n are replaced by $n\pi/\beta$ and for nontrivial solutions $n\geq 1$. Owing to the fact that the density distribution of Eq. (4.1) is somewhat rare, we shall not tabulate the corresponding frequencies in this paper.

5 Discussions and concluding remarks

We define an exact vibration solution as one which both the mode shape and the (characteristic) frequency equation can be expressed in an exact form. The solution should predict the full frequency spectrum. If only the axisymmetric modes are considered, the frequency spectrum would be incomplete.

We studied the circular and annular membranes with a radial power law density. The sector membranes are also considered for the first time. From the results for the lowest five frequencies, we can conclude the following.

In general, the sequence of modes may not correspond to increasing azimuthal mode number n. The normalized frequency increases with the absolute value of the power index $|\nu|$.

For the full circular membrane, the fundamental frequency occurs at $n=0$ where n is the number of nodal diameters. For the annular membrane, the frequency increases with respect to the inner radius b. When b is close to one, the width $1-b$ is the dominant factor and the differences in frequencies are small. For the sector membrane, $n-1$ is the number of internal radial nodes and the fundamental frequency occurs at $n=1$. Increased opening angle β increases the frequency. If one membrane has twice the opening angle of the other, then the frequencies of the larger membrane include those of the smaller membrane. The opposite, however, is not true.

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