Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 3, pp. 365-381

DOI: 10.4208/aamm.10-m11180
June 2012

Moving Finite Element Simulations for Reaction-Diffusion Systems

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Received 28 November 2011; Accepted (in revised version) 8 February 2012

Available online 30 April 2012

Abstract. This work is concerned with the numerical simulations for two reaction-diffusion systems, i.e., the Brusselator model and the Gray-Scott model. The numerical algorithm is based upon a moving finite element method which helps to resolve large solution gradients. High quality meshes are obtained for both the spot replication and the moving wave along boundaries by using proper monitor functions. Unlike [33], this work finds out the importance of the boundary grid redistribution which is particularly important for a class of problems for the Brusselator model. Several ways for verifying the quality of the numerical solutions are also proposed, which may be of important use for comparisons.

AMS subject classifications: 65M50, 65M60

Key words: Reaction-diffusion systems, Brusselator model, Gray-Scott model, moving finite element method.

1 Introduction

The generation of spatial pattern of tissue structures is one of the elementary processes in morphogenesis. Since the pioneering work of Turing [28] in 1952, there have been many studies on two-component reaction-diffusion systems for the formation of

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spatially complex patterns, see, e.g., [4,7–9,21]. A two-component reaction-diffusion system with general reaction terms f and g has the following form:

$$\frac{\partial u}{\partial t} - D_u \Delta u = f(u,v), \qquad \text{in } \Omega \times (0,\infty), \tag{1.1a}$$

$$\frac{\partial v}{\partial t} - D_v \Delta v = g(u,v), \qquad \text{in } \Omega \times (0,\infty), \tag{1.1b}$$

subject to the no-flux boundary conditions and initial conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.2a}$$

$$u(x,y,0) = u^0(x,y), \quad v(x,y,0) = v^0(x,y), \qquad \text{in } \Omega. \tag{1.2b}$$

Based on many specific forms of the reaction terms f and g, various models have been proposed for the pattern formation, such as Gray-Scott model [5], Schnakenberg model [25], Brusselator model [22] and Gierer-Meinhardt (GM) model [3]. These models have been widely used to model localization processes in nature, such as cell differentiation and morphogenesis [7,20], and the formation of sea-shell patterns [21].

Asymptotic and analytical methodologies for the analysis of these reaction-diffusion systems mainly concentrated on the one-dimensional model, see, e.g., [11, 12,27]. Recently, for the semi-strong interaction regime

$$D_u \ll 1 \quad \text{with } D_v = \mathcal{O}(1),$$

several asymptotic methodologies have been developed in two space dimensions, see, e.g., [30,31]. However, for the weak interaction regime, where $D_v = \mathcal{O}(D_u) \ll 1$, there are only a few works, see, e.g., [32]. Note that in [32], the case $D_u \ll 1$ and $D_v \gg 1$ is also considered. A survey of the asymptotic methods for reaction-diffusion systems is given in [29].

Numerical simulations play an important role in studying pattern formations, especially when there is difficulty in using the asymptotic and analytical approaches. In this paper, we will focus on the numerical studies on the two-dimensional Gray-Scott model and Brusselator model in the weak interaction regime. The Gray-Scott model is given by:

$$u_t = D_u \Delta u - uv^2 + \gamma(1-u), \tag{1.3a}$$

$$v_t = D_v \Delta v + uv^2 - (\gamma + \kappa)v, \tag{1.3b}$$

and the Brusselator model is given as follows:

$$u_t = D_u \Delta u + A + uv^2 - (B+1)u, \tag{1.4a}$$

$$v_t = D_v \Delta v + Bu - uv^2. \tag{1.4b}$$

Here, the diffusion coefficients $D_u, D_v \ll 1$ and f, k, A, B are chemical reaction parameters.