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Group Invariant Solutions of the Full Plastic Torsion of Rod with Arbitrary Shaped Cross Sections

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Abstract. Based on the theory of Lie group analysis, the full plastic torsion of rod with arbitrary shaped cross sections that consists in the equilibrium equation and the non-linear Saint Venant-Mises yield criterion is studied. Full symmetry group admitted by the equilibrium equation and the yield criterion is a finitely generated Lie group with ten parameters. Several subgroups of the full symmetry group are used to generate invariants and group invariant solutions. Moreover, physical explanations of each group invariant solution are discussed by all appropriate transformations. The methodology and solution techniques used belong to the analytical realm.

AMS subject classifications: 74C05, 76M60

Key words: Lie group analysis, group invariant solution, full plastic torsion, yield criterion.

1 Introduction

Lie group analysis is a very important tool in the study of invariant solutions of differential equation. It firstly appeared as an independent research in the Norwegian mathematician Lie's work [1]. The powerfulness of Lie group analysis has been extensively used to find analytic solutions of differential equations. Many scholars have made great efforts in this area [2–7].

In the theory of plasticity, Annin [7, 8] first solved the isothermal flow of ideally rigid-plastic medium with the von Mises yield criterion using Lie-Ovsiannikov group analysis, and he has also found a Lie group of point transformations that admitted by a system of equations of spatial flows. Some results of studying the group properties of equations of isothermal plastic flow of anisotropic and inhomogeneous media were obtained by Senashov [9]. Leonova [10] has studied group invariant solutions of

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equations of visco-plasticity and thermo-visco-plasticity. From the existing literature, we found the application of Lie group analysis is very wide [11–13].

Most classical literatures of solid mechanics [14, 15] describe the torsion of rod, but they are limited to discuss torsion of oval or rectangular rod. In this paper, we study on rod with arbitrary shaped cross sections. We obtain the symmetry group and classify the infinitesimal operators for the full plastic torsion of rod with arbitrary shaped cross sections.

2 Computational method of Lie group analysis

Let us now formulate the computational method of Lie group analysis. Suppose $x = (x_1, x_2, \cdots, x_n)$ is the independent variable, and $u = (u_1, u_2, \cdots, u_m)$ is the dependent variable. We consider the space M with coordinates x, u and the system of differential equations

$$F(F_1, F_2, \cdots, F_r) = 0, \qquad F_i = F_i(x, u, u^{(1)}, \cdots, u^{(s)}). \tag{2.1}$$

Here $u^{(k)}$ is totality of all partial derivatives of order k of functions u_α with respect to x_j, with $i \le r$, $k \le s$, $\alpha \le m$ and $j \le n$.

Infinitesimal operator is

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u_j} \begin{pmatrix} i = 1, \cdots, n \\ j = 1, \cdots, m \end{pmatrix}, \tag{2.2}$$

where ξ^i, η^j are tangent vector fields. X defines the group G. Then

$$XF|_M = 0, \tag{2.3}$$

where the notation $|_M$ means evaluated on M.

Symbol $p^\alpha_{i_1,\cdots,i_k}$ corresponds to the derivative

$$p^\alpha_{i_1,\cdots,i_k} = \frac{\partial^k u_\alpha}{\partial x_{i_1} \cdots \partial x_{i_n}}. \tag{2.4}$$

We consider the space $\widetilde{M}$ with coordinates x, u, $u^{(1)}$,$\cdots$, $u^{(k)}$. The k-th prolongation of infinitesimal operator is

$$X_k = X + \zeta^\alpha_i \partial_{p^\alpha_i} + \cdots + \zeta^\alpha_{i_1,\cdots,i_k}\partial_{p^\alpha_{i_1,\cdots,i_k}}, \tag{2.5a}$$

$$\zeta^\alpha_{i_1,\cdots,i_s} = D_{i_s}(\zeta^\alpha_{i_1\cdots i_{s-1}}) - p^\alpha_{\beta i_1\cdots i_{s-1}} D_{i_s}(\xi^\beta), \tag{2.5b}$$

$$D_{i_s} = \partial_{x_{i_s}} + p^\alpha_{i_s}\partial_{u_\alpha} + p^\alpha_{i_s j_1}\partial_{p^\alpha_{j_1}} + \cdots + p^\alpha_{i_s j_1\cdots j_{s-1}}\partial_{p^\alpha_{j_1\cdots j_{s-1}}}. \tag{2.5c}$$

Indices take the following values: $\alpha = 1 \cdots m$, $\beta, i, i_1 \cdots i_k = 1, \cdots, n$; $s \le k$. Repeated indices mean series sum over this index.Consequently, we can get

$$X_k F|_{\widetilde{M}} = 0. \tag{2.6}$$